ON THE FOURTH ORDER SCHRÖDINGER EQUATION IN THREE DIMENSIONS: DISPERSIVE ESTIMATES AND ZERO ENERGY RESONANCES

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Abstract. We study the fourth order Schrödinger operator $H = (-\Delta)^2 + V$ for a short range potential in three space dimensions. We provide a full classification of zero energy resonances and study the dynamic effect of each on the $L^1 \to L^\infty$ dispersive bounds. In all cases, we show that the natural $|t|^{-\frac{3}{4}}$ decay rate may be attained, though for some resonances this requires subtracting off a finite rank term, which we construct and analyze. The classification of these resonances, as well as their dynamical consequences differ from the Schrödinger operator $-\Delta + V$.

1. Introduction

We consider the linear fourth order Schrödinger equation in three spatial dimensions

$$i\psi_t = H\psi, \;\; \psi(0,x) = f(x), \;\; H := \Delta^2 + V, \;\; x \in \mathbb{R}^3.$$

Variants of this equation were introduced by Karpman [22] and Karpman and Shagalov [23] to account for small fourth-order dispersion in the propagation of laser beams in a bulk medium with Kerr nonlinearity, and may be used to model other "high dispersion" models. Linear dispersive estimates have recently been studied, [9, 16, 10], we continue this study to understand the structure and effect of zero energy resonances on the dynamics of the solution operator in the three dimensional case.

Fourth order Schrödinger equations have been studied in various contexts. For example, the stability and instability of solitary waves in a non-linear fourth order equation were considered in [26]. Well-posedness and scattering problems for various nonlinear fourth order equations have been studied by many authors, see for example [27, 28, 32, 33, 17, 18]. We note that time decay estimates we consider in this paper may be used in the study of special solutions to non-linear equations.

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In the free case, see [2], the solution operator $e^{-it\Delta^2}$ in d-dimensions preserves the L^2 norm and satisfies the following dispersive estimate

$$\|e^{-it\Delta^2} f\|_{L^\infty(\mathbb{R}^d)} \lesssim |t|^{-\frac{d}{4}} \|f\|_{L^1(\mathbb{R}^d)}.$$

In this paper we study the dispersive estimates in three spatial dimensions when there are obstructions at zero, i.e the distributional solutions to $H\psi = 0$ with $\psi \in L^\infty(\mathbb{R}^3)$. We provide a full classification of the zero energy obstructions as a finite dimensional space of eigenfunctions along with a ten-dimensional space of two distinct types of zero-energy resonances, see Section 7. As in the four dimensional case, [16], the zero energy obstructions in three dimensions have a more complicated structure than that of the Schrödinger operators $-\Delta + V$, [20, 8]. Let $P_{ac}(H)$ be the projection onto the absolutely continuous spectrum of H and $V(x)$ be a real-valued, polynomially decaying potential. We prove dispersive bounds of the form

$$\|e^{-itH} P_{ac}(H) f\|_{L^\infty} \lesssim |t|^{-\gamma} \|f\|_{L^1},$$

or a variant with spatial weights, for each type of zero energy obstruction where γ depends on the type of resonance. Such estimates can be used to study asymptotic stability of solitons for non-linear equations.

We introduce some notation to state our main results. We let $\langle \cdot \rangle = (1 + |\cdot|)^{\frac{1}{2}}$, and let $a-$ denote $a - \epsilon$ for a small, but fixed value of $\epsilon > 0$. We define the polynomially weighted L^p spaces,

$$L^{p,\sigma}(\mathbb{R}^3) := \{f \,:\, \langle \cdot \rangle^\sigma f \in L^p(\mathbb{R}^3)\}.$$

We provide a precise definition and characterization of resonances in Section 7 and Definition 4.2 below. We characterize the resonances in terms of distributional solutions to $H\psi = 0$. Heuristically, if $|\psi(x)| \sim 1$ as $|x| \to \infty$, we have a resonance of the first kind. If $|\psi(x)| \sim |x|^{-1}$ as $|x| \to \infty$ we have a resonance of the second kind, and if $|\psi(x)| \sim |x|^{-\frac{3}{2}-}$ we have a resonance of the third kind. The classification of the resonances in the fourth order Schrödinger equation requires a more detailed, subtle analysis than in the Schrödinger equation since the lower order terms in the expansion of Birman-Schwinger operator interact each other, see expansions of $M(\lambda)$ in Lemma 4.1. This causes complications in the classification of threshold obstructions which do not arise in the case of Schrödinger's equation or in the four dimensional case, see (20), (21), and Section 7. Our main results are summarized in the theorem below.

Theorem 1.1. *Let V be a real-valued potential satisfying $|V(x)| \lesssim \langle x \rangle^{-\beta}$ be such that there are no embedded eigenvalues in $[0,\infty)$ except possibly at zero. Then,*

i) If zero is regular, then if $\beta > 5$,

$$\|e^{-itH}P_{ac}(H)\|_{L^1\to L^\infty} \lesssim |t|^{-\frac{3}{4}}.$$

ii) If there is a resonance of the first kind at zero, then if $\beta > 7$,

$$\|e^{-itH}P_{ac}(H)\|_{L^1\to L^\infty} \lesssim |t|^{-\frac{3}{4}}.$$

iii) If there is a resonance of the second kind at zero, then if $\beta > 11$,

$$\|e^{-itH}P_{ac}(H)\|_{L^1\to L^\infty} \lesssim \begin{cases} |t|^{-\frac{3}{4}} & |t| \leq 1 \\ |t|^{-\frac{1}{4}} & |t| > 1 \end{cases}$$

Moreover, there is a time-dependent, finite-rank operator F_t satisfying $\|F_t\|_{L^1\to L^\infty} \lesssim \langle t \rangle^{-\frac{1}{4}}$ so that

$$\|e^{-itH}P_{ac}(H) - F_t\|_{L^1\to L^\infty} \lesssim |t|^{-\frac{3}{4}}.$$

iv) If there is a resonance of the third kind at zero, then if $\beta > 15$,

$$\|e^{-itH}P_{ac}(H)\|_{L^1\to L^\infty} \lesssim \begin{cases} |t|^{-\frac{3}{4}} & |t| \leq 1 \\ |t|^{-\frac{1}{4}} & |t| > 1 \end{cases}$$

Moreover, there is a time-dependent, finite-rank operator G_t satisfying $\|G_t\|_{L^1\to L^\infty} \lesssim \langle t \rangle^{-\frac{1}{4}}$ so that

$$\|e^{-itH}P_{ac}(H) - G_t\|_{L^1\to L^\infty} \lesssim |t|^{-\frac{1}{2}}.$$

Furthermore, one can improve this time decay at the cost of spatial weights,

$$\|e^{-itH}P_{ac}(H) - G_t\|_{L^{1,\frac{5}{2}}\to L^{\infty,-\frac{5}{2}}} \lesssim |t|^{-\frac{3}{4}}.$$

As in the two-dimensional Schrödinger equation and four-dimensional fourth order equation, we have a ‘mild’ type of resonance which does not affect the natural $|t|^{-\frac{d}{4}}$ decay rate. As in [9, 16, 10], we assume absence of positive eigenvalues. Under this assumption, a limiting absorption principle for H was established, see [9, Theorem 2.23], which we use to control the large energy portion of the evolution, which necessitates the larger bound as $t \to 0$. The large energy is unaffected by the zero energy obstructions, and our main contribution is to control the small energy portion of the evolution in all possible cases, which we show is bounded for all time and decays for large $|t|$.

In general, $|t|^{-\frac{d}{2}}$ decay rate for the Schrödinger evolution is affected by zero energy obstructions. In particular, the time decay for large $|t|$ is slower if there are obstructions at zero, see for example [21, 35, 34, 12, 7, 6, 13, 14]. It is natural to expect zero energy resonances to effect the time decay of the fourth order operator as well. This has been studied only in dimensions $d>3$; by Feng, Wu and Yao, [10], when $d>4$ as an operator between weighted L^2 spaces, and by the second and third authors when $d=4$, [16]. These works built on the work of Feng, Soffer and Yao in [9] which considered the case when zero is regular. This work in turn had its roots in Jensen and Kato's work [19], and [21] for $-\Delta+V$.

The free linear fourth order Schrödinger equation is studied by Ben-Artzi, Koch, and Saut [2]. They present sharp estimates on the derivatives of the kernel of the free operator, (including $\Delta^2\pm\Delta$). This followed work of Ben-Artzi and Nemirovsky which considered rather general operators of the form $f(-\Delta)+V$ on weighted L^2 spaces. Further generalized Schrödinger operators of the form $(-\Delta)^m+V$ were studied in [4], [11]. See also the work of Agmon [1] and Murata [29, 30, 31]. In particular, Murata's results for operators of the form $P(D)+V$ do not hold for the fourth order operator due to the degeneracy of $P(D)=\Delta^2$ at zero.

There are not many works considering the perturbed linear fourth order Schrödinger equation outside of the previously referenced recent works. There has been study of special solutions for nonlinear equations, see for example [24, 32, 33, 27, 28, 5]. See [25, 26] for a study of decay estimates for the fourth order wave equation.

Our results follow from careful expansions of the resolvent operators $(H-z)^{-1}$. We develop these expansions as perturbations of the free resolvent, for which, by using the second resolvent identity (see also [9]), we have the following representation:

$$R(H_0;z):=(\Delta^2-z)^{-1}=\frac{1}{2z^{\frac12}}\Big(R_0(z^{\frac12})-R_0(-z^{\frac12})\Big),\quad z\in\mathbb{C}\setminus[0,\infty). \tag{1}$$

Here $H_0=(-\Delta)^2$ and the R_0 is the Schrödinger resolvent $R_0(z^{\frac12}):=(-\Delta-z^{\frac12})^{-1}$. Since H_0 is essentially self-adjoint and $\sigma_{ac}(\Delta^2)=[0,\infty)$, by Weyl's criterion $\sigma_{ess}(H)=[0,\infty)$ for a sufficiently decaying potential. Let $\lambda\in\mathbb{R}^+$, we define the limiting resolvent operators by

$$R^{\pm}(H_0;\lambda):=R^{\pm}(H_0;\lambda\pm i0)=\lim_{\epsilon\to0^+}(\Delta^2-(\lambda\pm i\epsilon))^{-1}, \tag{2}$$

$$R_V^{\pm}(\lambda):=R_V^{\pm}(\lambda\pm i0)=\lim_{\epsilon\to0^+}(H-(\lambda\pm i\epsilon))^{-1}. \tag{3}$$

Note that using the representation (1) for $R(H_0;z)$ in definition (2) with $z=w^4$ for w in the first quandrant of the complex plane, and taking limits as $w\to\lambda$ and $w\to i\lambda$ in the first quadrant, we obtain

$$R^{\pm}(H_0;\lambda^4)=\frac{1}{2\lambda^2}\Big(R_0^{\pm}(\lambda^2)-R_0(-\lambda^2)\Big),\ \ \lambda>0. \tag{4}$$

Note that $R_0(-\lambda^2) : L^2 \to L^2$ since $-\Delta$ has nonnegative spectrum. Further, by Agmon's limiting absorption principle, [1], $R_0^\pm(\lambda^2)$ is well-defined between weighted L^2 spaces. Therefore, $R^\pm(H_0; \lambda^4)$ is also well-defined between these weighted spaces. This property is extended to $R_V^\pm(\lambda)$ in [9].

As usual, we use functional calculus and the Stone's formula to write

$$e^{-itH} P_{ac}(H) f(x) = \frac{1}{2\pi i} \int_0^\infty e^{-it\lambda} [R_V^+(\lambda) - R_V^-(\lambda)] f(x) d\lambda. \tag{5}$$

Here the difference of the perturbed resolvents provides the spectral measure. Our analysis in the three-dimensional case differs from the four dimensional case and previous works on the Schrödinger operator in several ways. First, the behavior of the free resolvents in (4) provides technical challenges in which various lower order terms in the expansions interact. These interactions complicate the inversion process as the operators whose kernels we study and need to invert are now the difference of different operators in the resolvent expansions, see (20) and (21) below. Such difficulties are new to this case, in the analysis of the Schrödinger resolvents, see [20], on can iterate the expansion procedure by examining the kernel of a single operator at each step. The techniques developed here may also be of use in dimensions $d = 1, 2$ or other high dispersion equations. Furthermore, the difference between the '+' and '-' resolvents in the Stone's formula, (5), which is crucial in the Schrödinger operators and the four-dimensional case, do not improve the analysis except in the most singular term in the case of a resonance of the third kind. Further, the classification of resonances differs from the Schrödinger case in several key aspects as shown in Section 7 below.

The paper is organized as follows. In Section 2 we provide definitions of the various notations we use to develop the operator expansions. In Section 3 we develop expansions for the free resolvent and establish the natural dispersive bound for the free operator. In Section 4 we develop expansions for the perturbed resolvent in a neighborhood of the threshold for each type of resonance that may occur. In Section 5 we utilize these expansions to prove the low energy version of Theorem 1.1. In Section 6 we prove the high energy version of Theorem 1.1. Finally, in Section 7 we provide a classification of the spectral subspaces associated to the different types of zero-energy obstructions.

2. Notation

For the convenience of the reader, we have gathered the notation and terminology we use throughout the paper.

For an operator $\mathcal{E}(\lambda)$, we write $\mathcal{E}(\lambda)=O_1(\lambda^{-\alpha})$ if it's kernel $\mathcal{E}(\lambda)(x,y)$ has the property

$$\sup_{x,y\in\mathbb{R}^3,\lambda>0}\big[\lambda^{\alpha}|\mathcal{E}(\lambda)(x,y)|+\lambda^{\alpha+1}|\partial_\lambda\mathcal{E}(\lambda)(x,y)|\big]<\infty. \tag{6}$$

Similarly, we use the notation $\mathcal{E}(\lambda)=O_1(\lambda^{-\alpha}g(x,y))$ if $\mathcal{E}(\lambda)(x,y)$ satisfies

$$|\mathcal{E}(\lambda)(x,y)|+\lambda|\partial_\lambda\mathcal{E}(\lambda)(x,y)|\lesssim\lambda^{-\alpha}g(x,y). \tag{7}$$

Recall the definition of the Hilbert-Schmidt norm of an operator K with kernel $K(x,y)$,

$$\|K\|_{HS}:=\bigg(\iint_{\mathbb{R}^6}|K(x,y)|^2\,dx\,dy\bigg)^{\frac12}.$$

We also recall the following terminology from [34, 7]:

Definition 2.1. *We say an operator $T:L^2(\mathbb{R}^2)\to L^2(\mathbb{R}^2)$ with kernel $T(\cdot,\cdot)$ is absolutely bounded if the operator with kernel $|T(\cdot,\cdot)|$ is bounded from $L^2(\mathbb{R}^2)$ to $L^2(\mathbb{R}^2)$.*

We note that Hilbert-Schmidt and finite-rank operators are absolutely bounded operators.

We will use the letter Γ to denote a generic absolutely bounded operator. In addition, Γ_θ denotes a λ dependent absolutely bounded operator satisfying

$$\big\||\Gamma_\theta|\big\|_{L^2\to L^2}+\lambda\big\||\partial_\lambda\Gamma_\theta|\big\|_{L^2\to L^2}\lesssim\lambda^{\theta},\quad \lambda>0. \tag{8}$$

The operator may vary depending on each occurrence and $\pm$ signs. The use of this notation allows us to significantly streamline the resolvent expansions developed in Section 4 as well as the proofs of the dispersive bounds in Section 5.

We use the smooth, even low energy cut-off χ defined by $\chi(\lambda)=1$ if $|\lambda|<\lambda_0\ll 1$ and $\chi(\lambda)=0$ when $|\lambda|>2\lambda_0$ for some sufficiently small constant $0<\lambda_0\ll 1$. In analyzing the high energy we utilize the complementary cut-off $\widetilde{\chi}(\lambda)=1-\chi(\lambda)$.

3. The Free Evolution

In this section we obtain expansions for the free fourth order Schrödinger resolvent operators $R^{\pm}(H_0;\lambda^4)$, using the identity (1) and the Bessel function representation of the Schrödinger free resolvents $R_0^{\pm}(\lambda^2)$. We use these expansions to establish dispersive estimates for the free fourth order Schrödinger evolution, and throughout the remainder of the paper to study the spectral measure for the perturbed operator.

Recall the expression of the free Schrödinger resolvents in dimension three, (see [15] for example)

$$R_0^{\pm}(\lambda^2)(x,y)=\frac{e^{\pm i\lambda|x-y|}}{4\pi|x-y|}.$$

Therefore, by (4),

$$R^{\pm}(H_0,\lambda^4)(x,y)=\frac{1}{2\lambda^2}\left(\frac{e^{\pm i\lambda|x-y|}}{4\pi|x-y|}-\frac{e^{-\lambda|x-y|}}{4\pi|x-y|}\right). \tag{9}$$

When, $\lambda|x-y|<1$, we have the following representation for the $R(H_0,\lambda^4)$

$$R^{\pm}(H_0,\lambda^4)(x,y)=\frac{a^{\pm}}{\lambda}+G_0+a_1^{\pm}\lambda G_1+a_3^{\pm}\lambda^3G_3+\lambda^4G_4+O(\lambda^5|x-y|^6). \tag{10}$$

Here

$$a^{\pm}:=\frac{1\pm i}{8\pi},\quad a_1^{\pm}=\frac{1\mp i}{8\pi\cdot(3!)},\quad a_3^{\pm}=\frac{1\pm i}{8\pi\cdot(5!)},\quad G_0(x,y)=-\frac{|x-y|}{8\pi}, \tag{11}$$

$$G_1(x,y)=|x-y|^2,\quad G_3(x,y)=|x-y|^4,\quad G_4(x,y)=-\frac{|x-y|^5}{4\pi\cdot(6!)}. \tag{12}$$

When $\lambda|x-y|\gtrsim 1$, the expansion remains valid. Notice that $G_0=(\Delta^2)^{-1}$.

The following lemma will be used repeatedly to obtain low energy dispersive estimates.

Lemma 3.1. *Fix $0<\alpha<4$. Assume that $\mathcal{E}(\lambda)=O_1(\lambda^{-\alpha})$ for $0<\lambda\lesssim 1$, then we have the bound*

$$\left|\int_0^{\infty}e^{it\lambda^4}\chi(\lambda)\lambda^3\mathcal{E}(\lambda)\,d\lambda\right|\lesssim\langle t\rangle^{-1+\frac{\alpha}{4}}. \tag{13}$$

Proof. By the support condition and since $\alpha<4$, the integral is bounded. Now, for $|t|>1$ we rewrite the integral in (13) as

$$\int_0^{t^{-\frac14}}e^{it\lambda^4}\lambda^3\chi(\lambda)\mathcal{E}(\lambda)\,d\lambda+\int_{t^{-\frac14}}^{\infty}e^{it\lambda^4}\lambda^3\chi(\lambda)\mathcal{E}(\lambda)\,d\lambda:=I+II.$$

We see that

$$|I|\le\int_0^{t^{-\frac14}}\lambda^{3-\alpha}\,d\lambda\lesssim t^{-1+\frac{\alpha}{4}}.$$

For the second term, we use $\partial_\lambda e^{it\lambda^4}/(4it)=e^{it\lambda^4}\lambda^3$ to integrate by parts once.

$$|II|\lesssim\frac{e^{it\lambda^4}\mathcal{E}(\lambda)}{4it}\Big|_{t^{-\frac14}}+\frac1t\int_{t^{-\frac14}}^{\infty}|\mathcal{E}'(\lambda)|\,d\lambda\lesssim t^{-1+\frac{\alpha}{4}}+\frac1t\int_{t^{-\frac14}}^{\infty}\lambda^{-\alpha-1}\,d\lambda\lesssim t^{-1+\frac{\alpha}{4}}.$$

□

Lemma 3.2. *We have the bound*

$$\sup_{x,y\in\mathbb{R}^3}\left|\int_0^{\infty}e^{it\lambda^4}\chi(\lambda)\lambda^3R^{\pm}(H_0,\lambda^4)(x,y)\,d\lambda\right|\lesssim\langle t\rangle^{-\frac34}.$$

Proof. Note that the cancellation between R^+ and R^- is not needed for, nor does it improve this bound. Using (9) we have

$$|R^{\pm}(H_0,\lambda^4)(x,y)| = \left|\frac{e^{\pm i\lambda|x-y|}-e^{-\lambda|x-y|}}{8\pi\lambda^2|x-y|}\right| \lesssim \frac{1}{\lambda}$$

uniformly in x, y for $\lambda|x-y|>1$. For $\lambda|x-y|<1$, we have

$$|R^{\pm}(H_0,\lambda^4)(x,y)| = \left|\frac{e^{\pm i\lambda|x-y|}-1+1-e^{-\lambda|x-y|}}{8\pi\lambda^2|x-y|}\right| \lesssim \frac{1}{\lambda}$$

by the mean value theorem. Similarly,

$$|\partial_\lambda R^{\pm}(H_0,\lambda^4)(x,y)| \lesssim \frac{1}{\lambda^2}$$

uniformly in x, y. Therefore

$$R^{\pm}(H_0,\lambda^4) = O_1(\lambda^{-1}), \tag{14}$$

and the claim follows from Lemma 3.1 with $\alpha = 1$. □

Remark 3.3. *The $t^{-\frac{3}{4}}$ bound is valid if we insert the high energy cutoff $\widetilde{\chi}(\lambda) = 1-\chi(\lambda)$ in place of the low energy cutoff $\chi(\lambda)$ in Lemma 3.1. However, the integral is not absolutely convergent, and is large for small $|t|$. That is,*

$$\left|\int_0^\infty e^{it\lambda^4}\widetilde{\chi}(\lambda)\lambda^3\mathcal{E}(\lambda)\,d\lambda\right| \lesssim |t|^{-1+\frac{\alpha}{4}}.$$

Consequently, we obtain the following estimate for the the free equation

$$\|e^{it\Delta^2}f\|_{L^1\to L^\infty} \lesssim t^{-\frac{3}{4}}.$$

4. Resolvent expansions near zero

In this section we provide the careful asymptotic expansions of the perturbed resolvent in a neighborhood of the threshold. To understand (5) for small energies, i.e. $\lambda \ll 1$, we use the symmetric resolvent identity. We define $U(x) = \text{sign}(V(x))$, $v(x) = |V(x)|^{\frac{1}{2}}$, and write

$$R_V^{\pm}(\lambda^4) = R^{\pm}(H_0,\lambda^4) - R^{\pm}(H_0,\lambda^4)v(M^{\pm}(\lambda))^{-1}vR^{\pm}(H_0,\lambda^4), \tag{15}$$

where $M^{\pm}(\lambda) = U + vR^{\pm}(H_0,\lambda^4)v$. As a result, we need to obtain expansions for $(M^{\pm}(\lambda))^{-1}$. The behavior of these operators as $\lambda \to 0$ depends on the type of resonances at zero energy, see Definition 4.2 below. We determine these expansions case by case and establish their contribution to spectral measure in Stone's formula, (5).

Let $T := U + vG_0v$, and recall (8), we have the following expansions.

Lemma 4.1. *For $0 < \lambda < 1$ define $M^{\pm}(\lambda) = U + vR^{\pm}(H_0, \lambda^4)v$. Let $P = v\langle \cdot, v\rangle \|V\|_1^{-1}$ denote the orthogonal projection onto the span of v. We have*

$$M^{\pm}(\lambda) = A^{\pm}(\lambda) + M_0^{\pm}(\lambda), \tag{16}$$

$$A^{\pm}(\lambda) = \frac{\|V\|_1 a^{\pm}}{\lambda} P + T, \tag{17}$$

where $T := U + vG_0v$ and $M_0^{\pm}(\lambda) = \Gamma_\ell$, for any $0 \le \ell \le 1$, provided that $v(x) \lesssim \langle x\rangle^{-\frac{5}{2}-\ell-}$. Moreover, for each $N = 1, 2, ...,$ and $\ell \in [0, 1]$,

$$M_0^{\pm}(\lambda) = \sum_{k=1}^{N} \lambda^k M_k^{\pm} + \Gamma_{N+\ell} \tag{18}$$

provided that $v(x) \lesssim \langle x\rangle^{-\frac{5}{2}-N-\ell-}$. Here the operators $M_k^{\pm}$ and the error term are Hilbert-Schmidt, and hence absolutely bounded operators. In particular

$$M_1^{\pm} = a_1^{\pm} vG_1v, \quad M_2^{\pm} = 0, \quad M_3^{\pm} = a_3^{\pm} vG_3v, \quad M_4^{\pm} = vG_4v, \tag{19}$$

where, the $a_j^{\pm}$'s and G_j's are defined in (11) *and* (12)*.*

Proof. We give a proof only for the case $N = 1, 2$, the other cases are similar. Using the expansion (10) for $\lambda|x-y| < 1$ and (9) for $\lambda|x-y| > 1$, we have

$$R^{\pm}(H_0, \lambda^4)(x, y) = \frac{a^{\pm}}{\lambda} + G_0 + a_1^{\pm}\lambda G_1 + O_1(\lambda^3|x-y|^4), \quad \lambda|x-y| < 1,$$

$$\begin{aligned} R^{\pm}(H_0, \lambda^4)(x, y) = \frac{a^{\pm}}{\lambda} + G_0 + a_1^{\pm}\lambda G_1 + \Big[\frac{e^{\pm i\lambda|x-y|} - e^{-\lambda|x-y|}}{8\pi\lambda^2|x-y|} - \frac{a^{\pm}}{\lambda} - G_0 - a_1^{\pm}\lambda G_1\Big] \\ = \frac{a^{\pm}}{\lambda} + G_0 + a_1^{\pm}\lambda G_1 + O_1(\lambda|x-y|^2), \quad \lambda|x-y| > 1. \end{aligned}$$

Using these in the definition of $M^{\pm}(\lambda)$ and $M_0^{\pm}(\lambda)$, we have

$$\Big|\big(M_0^{\pm}(\lambda) - a_1^{\pm}\lambda vG_1v\big)(x, y)\Big| \lesssim v(x)v(y)|x-y|^{\ell+2}\lambda^{\ell+1}, \ \ 0 \le \ell \le 2,$$

$$\Big|\partial_\lambda\big(M_0^{\pm}(\lambda) - a_1^{\pm}\lambda vG_1v\big)(x, y)\Big| \lesssim v(x)v(y)|x-y|^{\ell+2}\lambda^{\ell}, \ \ 0 \le \ell \le 2.$$

This yields the claim for $N = 1$ since the error term is an Hilbert-Schmidt operator if $v(x) \lesssim \langle x\rangle^{-\frac{5}{2}-1-\ell-}$. The case of $N = 2$ also follows since $M_2 = 0$ and $\ell \in [0, 2]$. □

The definition below classifies the type of resonances that may occur at the threshold energy. In Section 7, we establish this classification in detail. Since the free resolvent is unbounded as $\lambda \to 0$, this definition is somehow analogous to the definition of resonances from [20] and [34] for the two dimensional Schrödinger operators. However, there are important differences such as the appearance of the operators T_1, T_2 below. Specifically, the lower order terms in the expansions interact in such a way that T_1 and T_2 are now the differences of two separate operators. This phenomenon does not occur for the Schrödinger operators.

Definition 4.2. *i) Let $Q := \mathbb{1} - P$. We say that zero is regular point of the spectrum of $\Delta^2 + V$ provided QTQ is invertible on QL^2. In that case we define $D_0 := (QTQ)^{-1}$ as an absolutely bounded operator on QL^2, see Lemma 4.3 below.*

ii) Assume that zero is not regular point of the spectrum. Let S_1 be the Riesz projection onto the kernel of QTQ. Then $QTQ + S_1$ is invertible on QL^2. Accordingly, we define $D_0 = (QTQ + S_1)^{-1}$, as an operator on QL^2. This doesn't conflict with the previous definition since $S_1 = 0$ when zero is regular. We say there is a resonance of the first kind at zero if the operator

$$T_1 := S_1TPTS_1 - \frac{\|V\|_1}{3(8\pi)^2} S_1vG_1vS_1 \tag{20}$$

is invertible on S_1L^2.

iii) We say there is a resonance of the second kind if T_1 is not invertible on S_1L^2, but

$$T_2 := S_2vG_3vS_2 + \frac{10}{3} S_2vWvS_2 \tag{21}$$

is invertible. Here S_2 is the Riesz projection onto the kernel of T_1, and $W(x,y) = |x|^2|y|^2$. Moreover, we define $D_1 := (T_1 + S_2)^{-1}$ as an operator on S_1L^2.

iv) Finally if T_2 is not invertible we say there is a resonance of the third kind at zero. In this case the operator $T_3 := S_3vG_4vS_3$ is always invertible on S_3L^2 where S_3 the Riesz projection onto the kernel of T_2, see Lemma 7.6. We define $D_2 := (T_2 + S_3)^{-1}$ as an operator on S_3L^2.

As in the four dimensional operators, see the remarks after Definition 2.5 in [6] and after Definition 3.2 in [16], T is a compact perturbation of U. Hence, the Fredholm alternative guarantees that S_1 is a finite-rank projection. With these definitions first notice that, $S_3 \leq S_2 \leq S_1 \leq Q$, hence all S_j are finite-rank projections orthogonal to the span of v. Second, since T is a self-adjoint operator and S_1 is the Riesz projection onto its kernel, we have $S_1D_0 = D_0S_1 = S_1$. Similarly, $S_2D_1 = D_1S_2 = S_2$, $S_3D_2 = D_2S_3 = S_3$.

Lemma 4.3. *Let $|V(x)| \lesssim \langle x \rangle^{-\beta}$ for some $\beta > 5$, then QD_0Q is absolutely bounded.*

Proof. We prove the statement when $S_1 \neq 0$. We first assume that QUQ is invertible $QL^2 \to QL^2$. Using the resolvent identities, we have

$$QD_0Q = QUQ - QD_0Q(S_1 + vG_0v)QUQ = QUQ - S_1UQ - QD_0QvG_0vQUQ$$

Note that QUQ is absolutely bounded. Moreover, since S_1 is finite rank, any summand containing S_1 is finite rank, and hence absolutely bounded. For QD_0QvG_0vQUQ, we note vG_0v is an Hilbert-Schmidt operator for any $v(x) \lesssim \langle x\rangle^{-5/2-}$ and QD_0Q is bounded. Therefore, QD_0QvG_0v is Hilbert-Schmidt. Since the composition of absolutely bounded operators is absolutely bounded, QD_0Q is absolutely bounded.

If QUQ is not invertible, one can define π_0 as the Riesz projection onto the kernel of QUQ and see $QUQ+\pi_0$ is invertible on QL^2. Therefore, one can consider $Q[U+\pi_0+S_1+vG_0v-\pi_0]Q$ in the above argument to obtain the statement. □

Our aim in the rest of this section is to prove Theorem 4.4 below obtaining suitable expansions for $[M^{\pm}(\lambda)]^{-1}$ valid as $\lambda \to 0$ under the assumption that zero is regular and also in the cases when there are threshold obstructions. Recall the notation (8) and that the operators Γ_θ vary from line to line.

Theorem 4.4. *If zero is a regular point of the spectrum and if* $|v(x)| \lesssim \langle x\rangle^{-\frac{5}{2}-}$, *then*

$$[M^{\pm}(\lambda)]^{-1} = Q\Gamma_0Q + \Gamma_1.$$

If there is a resonance of the first kind at zero and if $|v(x)| \lesssim \langle x\rangle^{-\frac{7}{2}-}$, *then*

$$[M^{\pm}(\lambda)]^{-1} = Q\Gamma_{-1}Q + Q\Gamma_0 + \Gamma_0Q + \Gamma_1.$$

If there is a resonance of the second kind at zero and if $|v(x)| \lesssim \langle x\rangle^{-\frac{11}{2}-}$, *then*

$$[M^{\pm}(\lambda)]^{-1} = S_2\Gamma_{-3}S_2 + S_2\Gamma_{-2}Q + Q\Gamma_{-2}S_2 + S_2\Gamma_{-1} + \Gamma_{-1}S_2 + Q\Gamma_{-1}Q + Q\Gamma_0 + \Gamma_0Q + \Gamma_1.$$

If there is a resonance of the third kind at zero and if $|v(x)| \lesssim \langle x\rangle^{-\frac{15}{2}-}$, *then*

$$[M^{\pm}(\lambda)]^{-1} = \frac{1}{\lambda^4}S_3D_3S_3 + S_2\Gamma_{-3}S_2 + S_2\Gamma_{-2}Q + Q\Gamma_{-2}S_2 + S_2\Gamma_{-1} + \Gamma_{-1}S_2 + Q\Gamma_{-1}Q + Q\Gamma_0 + \Gamma_0Q + \Gamma_1.$$

Roughly speaking, modulo a finite rank term, the contribution to (5) of all of the operators in these expansions are of the same size with respect to the spectral parameter λ. We show in Lemma 5.1 that in the contribution to (15) having the operator Q on one side allows us to gain a power of λ, while having S_2 allows us to gain two powers of λ modulo the contribution of G_0.

Recall from (16) that $M^{\pm}(\lambda) = A^{\pm}(\lambda) + M_0^{\pm}(\lambda)$. If zero is regular then we have the following expansion for $(A^{\pm}(\lambda))^{-1}$.

Lemma 4.5. *Let $0 < \lambda \ll 1$. If zero is regular point of the spectrum of H. Then, we have*

$$(A^{\pm}(\lambda))^{-1} = QD_0Q + g^{\pm}(\lambda)S, \tag{22}$$

where $g^{\pm}(\lambda) = (\frac{a^{\pm}\|V\|_1}{\lambda} + c)^{-1}$ for some $c \in \mathbb{R}$, and

$$S = \begin{bmatrix} P & -PTQD_0Q \\ -QD_0QTP & QD_0QTPTQD_0Q \end{bmatrix}, \tag{23}$$

is a self-adjoint, finite rank operator.

Moreover, the same formula holds for $(A^{\pm}(\lambda) + S_1)^{-1}$ with $D_0 = (Q(T + S_1)Q)^{-1}$ if zero is not regular.

Proof. We prove the statement when $S_1 \neq 0$. The proof is identical in the regular case. Recalling (17), we write $A^{\pm}(\lambda) + S_1$ in the block format (using $PS_1 = S_1P = 0$):

$$A^{\pm}(\lambda) + S_1 = \begin{bmatrix} \frac{a^{\pm}\|V\|_1}{\lambda}P + PTP & PTQ \\ QTP & Q(T+S_1)Q \end{bmatrix} := \begin{bmatrix} a_{11} & a_{12} \\ a_{21} & a_{22} \end{bmatrix} \tag{24}$$

Since $Q(T+S_1)Q$ is invertible, by Feshbach formula (see, e.g., Lemma 2.8 in [7]) invertibility of $A^{\pm}(\lambda) + S_1$ hinges upon the existence of $d = (a_{11} - a_{12}a_{22}^{-1}a_{21})^{-1}$. Denoting $D_0 = (Q(T + S_1)Q)^{-1} : QL^2 \to QL^2$, we have

$$d = \big(\tfrac{a^{\pm}\|V\|_1}{\lambda}P + PTP - PTQD_0QTP\big)^{-1} = \big(\frac{a^{\pm}\|V\|_1}{\lambda} + c\big)^{-1}P =: g^{\pm}(\lambda)P$$

with $c = Tr(PTP - PTQD_0QTP) \in \mathbb{R}$. Therefore, d exists if λ is sufficiently small. Thus, by the Feshbach formula,

$$(A^{\pm}(\lambda) + S_1)^{-1} = \begin{bmatrix} d & -da_{12}a_{22}^{-1} \\ -a_{22}^{-1}a_{21}d & a_{22}^{-1}a_{21}da_{12}a_{22}^{-1} + a_{22}^{-1} \end{bmatrix} \tag{25}$$

$$= QD_0Q + g^{\pm}(\lambda)S. \tag{26}$$

□

Assume that $v(x) \lesssim \langle x\rangle^{-\frac{5}{2}-}$. Using (16), (18), the resolvent identity and Lemma 4.5 when zero is regular, we may write (for some $\epsilon > 0$)

$$[M^{\pm}]^{-1} = [A^{\pm} + M_0^{\pm}]^{-1} = [A^{\pm} + \Gamma_\epsilon]^{-1}$$
$$= [A^{\pm}]^{-1} - [A^{\pm}]^{-1}\Gamma_\epsilon[A^{\pm}]^{-1} + [A^{\pm}]^{-1}\Gamma_\epsilon[M^{\pm}]^{-1}\Gamma_\epsilon[A^{\pm}]^{-1} = Q\Gamma_0 Q + \Gamma_1,$$

proving Theorem 4.4 in the regular case.

Assuming that $v(x) \lesssim \langle x\rangle^{-\frac{7}{2}-}$, by Lemma 4.1, we have $M_0^{\pm} = \Gamma_1$. Also using (16) and Lemma 4.5 we obtain the following expansion in the case zero is not regular:

$$(M^{\pm}(\lambda) + S_1)^{-1} = (A^{\pm}(\lambda) + S_1 + M_0^{\pm}(\lambda))^{-1} \tag{27}$$
$$= (A^{\pm}(\lambda) + S_1)^{-1}\sum_{k=0}^{N}(-1)^k[M_0^{\pm}(\lambda)(A^{\pm}(\lambda) + S_1)^{-1}]^k + \Gamma_{N+1},\ N = 0, 1, \dots,$$
$$= QD_0Q + \Gamma_1.$$

The following lemma from [20] is the main tool to obtain the expansions of $M^{\pm}(\lambda)^{-1}$ when zero is not regular.

Lemma 4.6. *Let M be a closed operator on a Hilbert space $\mathcal{H}$ and S a projection. Suppose $M + S$ has a bounded inverse. Then M has a bounded inverse if and only if*

$$B := S - S(M + S)^{-1}S$$

has a bounded inverse in $S\mathcal{H}$, and in this case

$$M^{-1} = (M + S)^{-1} + (M + S)^{-1}SB^{-1}S(M + S)^{-1}. \tag{28}$$

We use this lemma with $M = M^{\pm}(\lambda)$ and $S = S_1$. Much of our technical work in the rest of this section is devoted to finding appropriate expansions for the inverse of $B^{\pm}(\lambda) = S_1 - S_1(M^{\pm}(\lambda) + S_1)^{-1}S_1$ on S_1L^2 under various spectral assumptions. For simplicity we work with + signs and drop the superscript.

We first list the orthogonality relations of various operators and projections we need.

$$S_iD_j = D_jS_i = S_i, i > j, \tag{29}$$

$$S_3 \leq S_2 \leq S_1 \leq Q = P^{\perp}, \tag{30}$$

$$S_1S = -S_1TP + S_1TPTQD_0Q,\ \ SS_1 = -PTS_1 + QD_0QTPTS_1, \tag{31}$$

$$S_1SS_1 = S_1TPTS_1, \tag{32}$$

$$SS_2 = S_2S = 0, \tag{33}$$

$$QM_1S_2 = S_2M_1Q = S_3M_1 = M_1S_3 = 0, \tag{34}$$

$$S_2 M_3 S_3 = S_3 M_3 S_2 = 0. \tag{35}$$

These can be checked using (23), (19), and $Qv = S_2TP = S_2vG_1vQ = S_2x_jv = S_3x_ix_jv = 0$, $i, j = 1, 2, 3$ (see Lemmas 7.4 and 7.5 below).

Using (18) with $N = 1$, $\ell = 0+$ and (26) in (27), and then using (32), we obtain

$$B(\lambda) = S_1 - S_1(M(\lambda) + S_1)^{-1}S_1 = -g(\lambda)S_1TPTS_1 + \lambda S_1M_1S_1 + \Gamma_{1+}, \tag{36}$$

provided that $|v(x)| \lesssim \langle x\rangle^{-\frac{7}{2}-}$.

Using (19), we have

$$\begin{aligned} g(\lambda)S_1TPTS_1 - \lambda S_1M_1S_1 &= g(\lambda)[S_1TPTS_1 - a_1\frac{\lambda}{g(\lambda)}S_1vG_1vS_1] \\ &= g(\lambda)T_1 - ca_1\lambda g(\lambda)S_1vG_1vS_1 = g(\lambda)T_1 + \Gamma_2, \end{aligned} \tag{37}$$

where

$$T_1 = S_1TPTS_1 - \frac{\|V\|_1}{3(8\pi)^2}S_1vG_1vS_1.$$

The second equality follows from

$$g(\lambda)[S_1TPTS_1 - a_1\frac{\lambda}{g(\lambda)}S_1vG_1vS_1] = g(\lambda)[S_1TPTS_1 - a_1(a\|V\|_1 + c\lambda)S_1vG_1vS_1],$$

and recalling the definitions of $g(\lambda)$, (11) and (19) to see

$$a^\pm a_1^\pm = \frac{(\pm i + 1)(\mp i + 1)}{(8\pi)^2(3!)} = \frac{2}{(8\pi)^2(3!)} = \frac{1}{3(8\pi)^2}.$$

In the case when there is a resonance of the first kind at zero, namely when T_1 is invertible, using (37) in (36), we obtain

$$B(\lambda)^{-1} = (-g(\lambda)T_1 + \Gamma_{1+})^{-1} = \Gamma_{-1},$$

provided that $v(x) \lesssim \langle x\rangle^{-\frac{7}{2}-}$. Using this and (27) in (28), we obtain

$$\begin{aligned} [M(\lambda)]^{-1} &= QD_0Q + \Gamma_1 + (QD_0Q + \Gamma_1)S_1\Gamma_{-1}S_1(QD_0Q + \Gamma_1) \\ &= Q\Gamma_{-1}Q + Q\Gamma_0 + \Gamma_0Q + \Gamma_1, \end{aligned}$$

proving Theorem 4.4 in the case when there is a resonance of the first kind.

In the case when there is a resonance of the second or third kind, namely when $S_2 \neq 0$, we need more detailed expansions for $B(\lambda)$, and hence for $(M(\lambda) + S_1)^{-1}$.

Using (18) and (19) in (27) we obtain

$$(M(\lambda) + S_1)^{-1} = QD_0Q + g^\pm(\lambda)S - \lambda QD_0QM_1QD_0Q \tag{38}$$

$$- \lambda g^{\pm}(\lambda)\big[QD_0QM_1S + SM_1QD_0Q\big] + \lambda^2 QD_0Q(M_1QD_0Q)^2$$
$$- \lambda(g^{\pm}(\lambda))^2 SM_1S + \lambda^2 g^{\pm}(\lambda)\big[S(M_1QD_0Q)^2 + QD_0QM_1SM_1QD_0Q + (QD_0QM_1)^2S\big]$$
$$- \lambda^3 QD_0Q\big[M_3 + M_1(QD_0QM_1)^2\big]QD_0Q + \Gamma_{3+},$$

provided that $v(x) \lesssim \langle x\rangle^{-\frac{11}{2}-}$.

Using (29)-(32) and (37) in (38), we obtain

$$S_1(M(\lambda)+S_1)^{-1}S_1 = S_1 + g(\lambda)T_1 \tag{39}$$
$$- ca_1\lambda g(\lambda)S_1vG_1vS_1 - \lambda g(\lambda)S_1\big[M_1S + SM_1\big]S_1 + \lambda^2 S_1M_1QD_0QM_1S_1$$
$$- \lambda(g(\lambda))^2 S_1SM_1SS_1 + \lambda^2 g(\lambda)\big[S_1SM_1QD_0QM_1S + S_1M_1SM_1S_1 + S_1M_1QD_0QM_1SS_1\big]$$
$$- \lambda^3 S_1\big[M_3 + M_1(QD_0QM_1)^2\big]S_1 + \Gamma_{3+}.$$

Therefore

$$B(\lambda) = S_1 - S_1(M(\lambda)+S_1)^{-1}S_1 = -g(\lambda)T_1$$
$$+ a_1\lambda g(\lambda)S_1(cM_1 + SM_1 + M_1S)S_1 - \lambda^2 S_1M_1QD_0QM_1S_1$$
$$+ \lambda(g(\lambda))^2 S_1SM_1SS_1 + \lambda^2 g(\lambda)\big[S_1SM_1QD_0QM_1S + S_1M_1SM_1S_1 + S_1M_1QD_0QM_1SS_1\big]$$
$$+ \lambda^3 S_1\big[M_3 + M_1(QD_0QM_1)^2\big]S_1 + \Gamma_{3+}.$$

Let $U_1 = S_1 - S_2$, $U_2 = S_2 - S_3$, and $U = U_1 + U_2$. In block form, we have

$$B(\lambda) = \begin{bmatrix} S_3B(\lambda)S_3 & S_3B(\lambda)U \\ UB(\lambda)S_3 & UB(\lambda)U \end{bmatrix}, \tag{40}$$

$$UB(\lambda)U = \begin{bmatrix} U_2B(\lambda)U_2 & U_2B(\lambda)U_1 \\ U_1B(\lambda)U_2 & U_1B(\lambda)U_1 \end{bmatrix}. \tag{41}$$

We first invert $UB(\lambda)U$ for small λ. We have

$$U_1B(\lambda)U_1 = -g(\lambda)U_1T_1U_1 + U_1\Gamma_2U_1,$$

Using (33) and (34), we obtain

$$U_1B(\lambda)U_2 = a_1\lambda g(\lambda)U_1SvG_1vU_2 + U_1\Gamma_3U_2 = -a_1\lambda g(\lambda)U_1TPvG_1vU_2 + U_1\Gamma_3U_2,$$

Similarly,

$$U_2B(\lambda)U_1 = -a_1\lambda g(\lambda)U_2vG_1vPTU_1 + U_2\Gamma_3U_1,$$

and

$$U_2B(\lambda)U_2 = \lambda^3 U_2M_3U_2 - \lambda^2 g(\lambda)U_2M_1SM_1U_2 + \Gamma_4$$
$$= a_3\lambda^3 U_2vG_3vU_2 - a_1^2\lambda^2 g(\lambda)U_2vG_1vSvG_1vU_2 + U_2\Gamma_{3+}U_2.$$

Note that by (34)

$$U_2vG_1vSvG_1vU_2 = U_2vG_1vPvG_1vU_2 = \|V\|_{L^1}U_2vWvU_2,$$

where $W(x,y) = |x|^2|y|^2$. In the second equality we used $G_1(x,y) = |x|^2 - 2x\cdot y + |y|^2$ and $S_2x_jv = S_2v = 0$. Also noting that

$$\frac{a_3\lambda}{g(\lambda)} = a_3a\|V\|_1 + ca_3\lambda = \frac{2i\|V\|_1}{5!(8\pi)^2} + ca_3\lambda, \quad a_1^2 = \frac{-2i}{(3!)^2(8\pi)^2},$$

we obtain

$$U_2B(\lambda)U_2 = \frac{2i}{5!(8\pi)^2}\|V\|_{L^1}\lambda^2 g(\lambda)[U_2vG_3vU_2 + \frac{10}{3}U_2vWvU_2] + \Gamma_4$$
$$= \frac{2i}{5!(8\pi)^2}\|V\|_{L^1}\lambda^2 g(\lambda)U_2T_2U_2 + U_2\Gamma_{3+}U_2.$$

If $U_1 = 0$, i.e. $S_1 = S_2$, then we can invert UBU as

$$(UB(\lambda)U)^{-1} = \frac{5!(8\pi)^2}{2i\|V\|_{L^1}\lambda^2 g(\lambda)}(U_2T_2U_2)^{-1} + U_2\Gamma_{-3+}U_2 = U_2\Gamma_{-3}U_2.$$

If $U_1 \neq 0$, we invert $UB(\lambda)U$ using Feshbach's formula. Note that, we can rewrite (41) using the calculations above:

$$UB(\lambda)U = -g(\lambda)\begin{bmatrix} -\frac{2i}{5!(8\pi)^2}\|V\|_{L^1}\lambda^2 U_2T_2U_2 + U_2\Gamma_{2+}U_2 & a_1\lambda U_2vG_1vPTU_1 + U_2\Gamma_2U_1 \\ a_1\lambda U_1TPvG_1vU_2 + U_1\Gamma_2U_2 & U_1T_1U_1 + U_1\Gamma_1U_1 \end{bmatrix}.$$

Note that a_{22} is invertible. Therefore $UB(\lambda)U$ is invertible provided the following exists

$$d = \Big(-\frac{2i\|V\|_{L^1}}{5!(8\pi)^2}\lambda^2 U_2T_2U_2 - a_1^2\lambda^2 S_2vG_1vPTU_1(U_1T_1U_1)^{-1}U_1TPvG_1vU_2 + U_2\Gamma_{2+}U_2\Big)^{-1}$$
$$= -\frac{5!(8\pi)^2}{2i\|V\|_{L^1}\lambda^2}\Big(U_2T_2U_2 - \frac{10}{3\|V\|_{L^1}}U_2vG_1vPTU_1(U_1T_1U_1)^{-1}U_1TPvG_1vU_2\Big)^{-1} + U_2\Gamma_{-2+}U_2.$$

Note that, since $S_2v = S_2x_jv = 0$ we can rewrite the operator in parenthesis as

$$U_2T_2U_2 - \frac{10\langle (U_1T_1U_1)^{-1}U_1Tv, U_1Tv\rangle}{3\|V\|_{L^1}}U_2vWvU_2$$
$$= U_2vG_3vU_2 + \frac{10}{3}\Big(1 - \frac{\langle (U_1T_1U_1)^{-1}U_1Tv, U_1Tv\rangle}{\|V\|_{L^1}}\Big)U_2vWvU_2.$$

Note that by Lemma 7.5 below the kernel of T_2 agrees with the kernel of $S_2vG_3vS_2$. Therefore $U_2vG_3vU_2$ is invertible and positive definite. Since U_2vWvU_2 is positive semi-definite, the inverse exists if we can prove that $\langle (U_1T_1U_1)^{-1}U_1Tv, U_1Tv\rangle \leq \|V\|_{L^1}$. Note that

$$U_1TPTU_1u = \frac{1}{\|V\|_{L^1}}U_1(Tv)\langle u, U_1(Tv)\rangle.$$

Also note that $U_1T_1U_1 - U_1TPTU_1$ is positive semi-definite. Therefore the required bound follows from the following lemma with $\mathcal{H} = U_1L^2$, $z = U_1(Tv)$, $\alpha = \frac{1}{\|V\|_{L^1}}$, and $\mathcal{S} = U_1T_1U_1 - U_1TPTU_1$.

Lemma 4.7. *Let $\mathcal{H}$ be a Hilbert space. Fix $z \in \mathcal{H}$ and $\alpha > 0$ and let $\mathcal{T}(u) = \alpha z\langle u, z\rangle$, $u \in \mathcal{H}$. Let $\mathcal{S}$ be a positive semi-definite operator on $\mathcal{H}$ so that $\mathcal{T} + \mathcal{S}$ is invertible. Then,*

$$0 \leq \big\langle (\mathcal{T}+\mathcal{S})^{-1}z, z\big\rangle \leq \frac{1}{\alpha}.$$

Proof. Let $w = (\mathcal{T}+\mathcal{S})^{-1}z$. We have

$$z = \mathcal{T}w + \mathcal{S}w = \alpha z\langle w, z\rangle + \mathcal{S}w, \text{ and hence } \mathcal{S}w = z - \alpha z\langle w, z\rangle.$$

Then since $\mathcal{S}$ is positive semi-definite,

$$0 \leq \langle \mathcal{S}w, w\rangle = \big\langle z - \alpha z\langle w, z\rangle, w\big\rangle = \langle z, w\rangle - \alpha|\langle z, w\rangle|^2.$$

Therefore, $\langle (\mathcal{T}+\mathcal{S})^{-1}z, z\rangle = \langle w, z\rangle \in \mathbb{R}$ and

$$0 \leq \langle w, z\rangle \leq \frac{1}{\alpha}. \qquad \square$$

We conclude that

$$d = \lambda^{-2}U_2DU_2 + U_2\Gamma_{-2+}U_2 = U_2\Gamma_{-2}U_2.$$

Using this in the Feshbach formula (25), we obtain

$$(UB(\lambda)U)^{-1} = -\frac{1}{g(\lambda)}\begin{bmatrix} U_2\Gamma_{-2}U_2 + U_2\Gamma_{-1}U_2 & U_2\Gamma_{-1}U_1 \\ U_1\Gamma_{-1}U_2 & U_1\Gamma_0U_1 \end{bmatrix} \tag{42}$$

$$= U_2\Gamma_{-3}U_2 + U_2\Gamma_{-2}U_1 + U_1\Gamma_{-2}U_2 + U_1\Gamma_{-1}U_1.$$

We now focus on the case $S_3 = 0$, $U_2 = S_2 \neq 0$. We have $B(\lambda)^{-1} = (UB(\lambda)U)^{-1}$. Using (38) and orthogonality relations (29)-(34), we have

$$S_2(M(\lambda)+S_1)^{-1} = (M(\lambda)+S_1)^{-1}S_2 = S_2 + \Gamma_2.$$

Also recall that

$$(M(\lambda)+S_1)^{-1} = QD_0Q + \Gamma_1.$$

Using these in (28), we have

$$
\begin{aligned}
(43)\quad M(\lambda)^{-1} &= QD_0Q + \Gamma_1 \\
&\quad + (M(\lambda)+S_1)^{-1}\big[U_2\Gamma_{-3}U_2 + U_2\Gamma_{-2}U_1 + U_1\Gamma_{-2}U_2 + U_1\Gamma_{-1}U_1\big](M(\lambda)+S_1)^{-1} \\
&= S_2\Gamma_{-3}S_2 + S_2\Gamma_{-2}Q + Q\Gamma_{-2}S_2 + S_2\Gamma_{-1} + \Gamma_{-1}S_2 + Q\Gamma_{-1}Q + Q\Gamma_0 + \Gamma_0 Q + \Gamma_1.
\end{aligned}
$$

This expansion is valid also in the case $U_1 = 0$, proving Theorem 4.4 in the case of resonance of the second kind.

We consider the final case, when $S_3 \neq 0$. Using

$$(A^{\pm}(\lambda)+S_1)^{-1}S_3 = S_3(A^{\pm}(\lambda)+S_1)^{-1} = S_3,$$

$$S_3M_0^{\pm} = M_0^{\pm}S_3 = \Gamma_3,$$

$$S_3M_0^{\pm}S_3 = \lambda^4 S_3vG_4vS_3 + \Gamma_5 = \lambda^4 T_3 + \Gamma_5,$$

we have

$$S_3B^{\pm}S_3 = -S_3\sum_{k=1}^{4}(-1)^k[M_0^{\pm}(\lambda)(A^{\pm}(\lambda)+S_1)^{-1}]^k S_3 + \Gamma_5 = \lambda^4 T_3 + \Gamma_5,$$

provided that $v(x) \lesssim \langle x\rangle^{-\frac{15}{2}-}$.

If $U \neq 0$, we invert $B(\lambda)$ using Feshbach's formula for the block form (40). Note that

$$d = \big(S_3BS_3 - S_3BU(UBU)^{-1}UBS_3\big)^{-1}.$$

The leading term is $\lambda^4 T_3 + \Gamma_5$. We write the second term as

$$
\begin{aligned}
&S_3BU_2(UBU)^{-1}U_2BS_3 + S_3BU_1(UBU)^{-1}U_2BS_3 \\
&\qquad + S_3BU_2(UBU)^{-1}U_1BS_3 + S_3BU_1(UBU)^{-1}U_1BS_3 = \Gamma_5.
\end{aligned}
$$

To obtain the estimate, we used $S_3BU_2 = \Gamma_4$, $S_3BU_1 = \Gamma_3$, and (42). Therefore, for small $\lambda > 0$,

$$d = \lambda^{-4}D_3 + S_3\Gamma_{-3}S_3 = S_3\Gamma_{-4}S_3.$$

Using this in Feshbach's formula for the block form (40) we obtain

$$
\begin{aligned}
B(\lambda)^{-1} &= \lambda^{-4}D_3 + S_3\Gamma_{-3}S_3 + S_3\Gamma_{-4}S_3BU(UBU)^{-1} + (UBU)^{-1}UBS_3\Gamma_{-4}S_3 \\
&\qquad + \lambda^{-4}(UBU)^{-1}UBS_3\Gamma_{-4}S_3BU(UBU)^{-1} + (UBU)^{-1}.
\end{aligned}
$$

Using (42), decomposing $U = U_1 + U_2$ as above, and using $S_3 B U_2 = \Gamma_4$, $S_3 B U_1 = \Gamma_3$, we have

$$B(\lambda)^{-1} = \lambda^{-4} D_3 + S_2\Gamma_{-3}S_2 + S_2\Gamma_{-2}S_1 + S_1\Gamma_{-2}S_2 + S_1\Gamma_{-1}S_1.$$

Finally, using

$$S_2(M(\lambda)+S_1)^{-1} = (M(\lambda)+S_1)^{-1}S_2 = S_2 + \Gamma_2,$$

$$S_3(M(\lambda)+S_1)^{-1} = (M(\lambda)+S_1)^{-1}S_3 = S_3 + \Gamma_3,$$

$$(M(\lambda)+S_1)^{-1} = QD_0Q + \Gamma_1,$$

we obtain Theorem 4.4 in the case of a resonance of the third kind.

5. Low energy dispersive estimates

In this section we analyze the perturbed evolution e^{-itH} in $L^1 \to L^\infty$ setting for small energy, when the spectral variable λ is in a small neighborhood of the threshold energy $\lambda = 0$. As in the free case, we represent the solution via Stone's formula, (5). As usual, we analyze (5) separately for large energy, when $\lambda \gtrsim 1$, and for small energy, when $\lambda \ll 1$, see for example [34, 7]. The effect of the presence of zero energy resonances is only felt in the small energy regime. Different resonances change the asymptotic behavior of the perturbed resolvents and hence that of the spectral measure as $\lambda \to 0$ which we study in this section. The large energy argument appears in Section 6 to complete the proof of Theorem 1.1.

We start with the following lemma which will be used repeatedly.

Lemma 5.1. *Assume that $v(x) \lesssim \langle x\rangle^{-\frac{5}{2}-}$, then*

$$\sup_y \big\|[QvR^{\pm}(H_0,\lambda^4)](\cdot,y)\big\|_{L^2} \lesssim 1, \quad \textit{and} \quad \sup_y \big\|\partial_\lambda[QvR^{\pm}(H_0,\lambda^4)](\cdot,y)\big\|_{L^2} \lesssim \frac{1}{\lambda}.$$

Assuming that $v(x) \lesssim \langle x\rangle^{-\frac{7}{2}-}$, we have

$$\sup_y \big\|\big[S_2v(R^{\pm}(H_0,\lambda^4)-G_0)\big](\cdot,y)\big\|_{L^2} \lesssim \lambda, \quad \sup_y \big\|\partial_\lambda\big[S_2v(R^{\pm}(H_0,\lambda^4)-G_0)\big](\cdot,y)\big\|_{L^2} \lesssim 1,$$

and

$$\sup_y \big\|\big[S_2v(R^{+}(H_0,\lambda^4)-R^{-}(H_0,\lambda^4))\big](\cdot,y)\big\|_{L^2} \lesssim \lambda,$$

$$\sup_y \big\|\partial_\lambda\big[S_2v(R^{+}(H_0,\lambda^4)-R^{-}(H_0,\lambda^4))\big](\cdot,y)\big\|_{L^2} \lesssim 1.$$

Proof. We prove the assertion for + sign. Recall the expansion (10). Using the fact $Qv=0$ we have

$$[QvR^+(H_0,\lambda^4)](y_2,y) = \frac{1}{8\pi\lambda}\int_{\mathbb{R}^3} Q(y_2,y_1)v(y_1)[F(\lambda|y-y_1|)-F(\lambda|y|)]dy_1$$
$$= \frac{1}{8\pi}\int_{\mathbb{R}^3} Q(y_2,y_1)v(y_1)\int_{|y|}^{|y-y_1|} F'(\lambda s)dsdy_1,$$

where

$$F(p) = \frac{e^{ip}-e^{-p}}{p}.$$

Noting that $|F'(p)|\lesssim 1$, and using the absolute boundedness of Q, we obtain

$$\big\|[QvR^+(H_0,\lambda^4)](\cdot,y)\big\|_{L^2} \lesssim \Big\|\int_{\mathbb{R}^3}|Q(y_2,y_1)||v(y_1)|\langle y_1\rangle dy_1\Big\|_{L^2_{y_2}} \lesssim \|v(y_1)\langle y_1\rangle\|_{L^2}\lesssim 1,$$

uniformly in y.

Now consider $S_2v(R^+(H_0,\lambda^4)-G_0)$. We have

$$[S_2v(R^+(H_0,\lambda^4)-G_0)](y_2,y) = \frac{1}{8\pi\lambda}\int_{\mathbb{R}^3} S_2(y_2,y_1)v(y_1)F(\lambda|y-y_1|)dy_1$$

where

$$F(p) = \frac{e^{ip}-e^{-p}}{p}+p.$$

Noting that $S_2v=0$ we can rewrite the integral above as

$$\frac{1}{8\pi\lambda}\int_{\mathbb{R}^3} S_2(y_2,y_1)v(y_1)[F(\lambda|y-y_1|)-F(\lambda|y|)]dy_1$$
$$= \frac{1}{8\pi}\int_{\mathbb{R}^3} S_2(y_2,y_1)v(y_1)\int_{|y|}^{|y-y_1|}F'(\lambda s)dsdy_1.$$

Furthermore, one has $S_2y_jv=0$, and hence

$$\int_{\mathbb{R}^3} S_2(y_2,y_1)v(y_1)\frac{y_1\cdot y}{|y|}F'(\lambda|y|)dy_1 = \int_{\mathbb{R}^3} S_2(y_2,y_1)v(y_1)y_1dy_1\cdot\frac{y}{|y|}F'(\lambda|y|)=0.$$

This gives

$$\int_{\mathbb{R}^3} S_2(y_2,y_1)v(y_1)\int_{|y|}^{|y-y_1|}F'(\lambda s)dsdy_1 \tag{44}$$
$$= \int_{\mathbb{R}^3} S_2(y_2,y_1)v(y_1)\Big[\int_{|y|}^{|y-y_1|}F'(\lambda s)ds+\frac{y_1\cdot y}{|y|}F'(\lambda|y|)\Big]dy_1$$
$$= \int_{\mathbb{R}^3} S_2(y_2,y_1)v(y_1)\left[\int_{|y|-\frac{y_1\cdot y}{|y|}}^{|y-y_1|}F'(\lambda s)ds - \int_{|y|-\frac{y_1\cdot y}{|y|}}^{|y|}F'(\lambda s)ds + \int_{|y|-\frac{y_1\cdot y}{|y|}}^{|y|}F'(\lambda|y|)ds\right]dy_1$$

$$= \int_{\mathbb{R}^3} S_2(y_2, y_1) v(y_1) \Big[\int_{|y| - \frac{y_1 \cdot y}{|y|}}^{|y-y_1|} F'(\lambda s) ds + \lambda \int_{|y| - \frac{y_1 \cdot y}{|y|}}^{|y|} \int_s^{|y|} F''(\lambda k) dk ds \Big] dy_1.$$

To control the integrals in (44) notice that $|F^{(k)}(p)| \lesssim p^{2-k}$ for $k = 1, 2$. Therefore, for $|y| - \left|\frac{y_1 \cdot y}{|y|}\right| \geq 0$, we obtain

$$\Big| \int_{|y| - \frac{y_1 \cdot y}{|y|}}^{|y-y_1|} F'(\lambda s) ds \Big| \lesssim \lambda \Big| \int_{|y| - \frac{y_1 \cdot y}{|y|}}^{|y-y_1|} s ds \Big| \lesssim \lambda \Big| |y - y_1|^2 - (|y| - \frac{y_1 \cdot y}{|y|})^2 \Big| \lesssim \lambda \langle y_1 \rangle^2. \tag{45}$$

Note that if $|y| - \left|\frac{y_1 \cdot y}{|y|}\right| < 0$, one has $|y|, |y - y_1| < |y_1|$ and therefore the above inequality is trivial.

For the second term in (44), we have

$$\int_{|y| - \frac{y_1 \cdot y}{|y|}}^{|y|} \int_s^{|y|} F''(\lambda k) dk ds = \int_{|y| - \frac{y_1 \cdot y}{|y|}}^{|y|} [k - |y| + \frac{y_1 \cdot y}{|y|}] F''(\lambda k) dk. \tag{46}$$

Noting that $|[k - |y| + \frac{y_1 \cdot y}{|y|}]| \lesssim \langle y_1 \rangle$ and $|F''(\lambda k)| \lesssim 1$. This term can be controlled by $\langle y_1 \rangle^2$. Finally, by (45) and (46), we obtain

$$\begin{aligned} &\big\| [S_2 v(R^+(H_0, \lambda^4) - G_0)](\cdot, y) \big\|_{L^2} \\ &\qquad \lesssim \lambda \Big\| \int_{\mathbb{R}^3} |S_2(y_2, y_1)| |v(y_1)| \langle y_1 \rangle^2 dy_1 \Big\|_{L^2_{y_2}} \lesssim \lambda \| v(y_1) \langle y_1 \rangle^2 \|_{L^2} \lesssim \lambda, \end{aligned}$$

uniformly in y.

To establish the bound on the first derivative, note that

$$\partial_\lambda F(\lambda r) = \frac{1}{\lambda} \Big[\frac{[i(\lambda r) - 1] e^{i(\lambda r)} + e^{-(\lambda r)} [(\lambda r) + 1]}{(\lambda r)} - (\lambda r) \Big] =: \frac{1}{\lambda} \tilde{F}(\lambda r)$$

Since one has $|\tilde{F}^k(p)| \lesssim p^{2-k}$, one can apply the same method to $\tilde{F}$ to finish the proof.

The last assertion follows from noting that the bounds used on $S_2 v(R^\pm(H_0, \lambda^4) - G_0)$ also apply to $S_2 v(R^+(H_0, \lambda^4) - R^-(H_0, \lambda^4))$, see (10) and the subsequent discussion. □

We first consider the case when zero is regular ($S_1 = 0$) or when there is a resonance of the first kind $S_1 \neq 0, S_2 = 0$.

Theorem 5.2. *Assume that $v(x) \lesssim \langle x \rangle^{-\frac{5}{2}-}$ and $S_1 = 0$, or that $v(x) \lesssim \langle x \rangle^{-\frac{7}{2}-}$ and $S_1 \neq 0, S_2 = 0$. Then*

$$\sup_{x, y \in \mathbb{R}^3} \Big| \int_0^\infty e^{it\lambda^4} \lambda^3 \chi(\lambda) R_V^\pm(\lambda^4)(x, y) \, d\lambda \Big| \lesssim \langle t \rangle^{-\frac{3}{4}}. \tag{47}$$

Proof. Recall (15):

$$R_V^{\pm}(\lambda^4) = R^{\pm}(H_0,\lambda^4) - R^{\pm}(H_0,\lambda^4)v(M^{\pm}(\lambda))^{-1}vR^{\pm}(H_0,\lambda^4).$$

We already obtained the required bound for the free term in Lemma 3.2. For the correction term, dropping the $\pm$ signs, the claim will follow from Lemma 3.1 with

$$\mathcal{E}(\lambda)(x,y) = \big[R(H_0,\lambda^4)v(M(\lambda))^{-1}vR(H_0,\lambda^4)\big](x,y). \tag{48}$$

By Theorem 4.4, in the regular case we have $M(\lambda)^{-1} = QD_0Q+\Gamma_1$. In the case of a resonance of the first kind, we have

$$M(\lambda)^{-1} = Q\Gamma_{-1}Q + Q\Gamma_0 + \Gamma_0 Q + \Gamma_1.$$

First consider the contribution of Γ_1 to (48):

$$\big[R(H_0,\lambda^4)v\Gamma_1 vR(H_0,\lambda^4)\big](x,y).$$

Note that, by (14) we have

$$\|vR(H_0,\lambda^4)(\cdot,y)\|_{L^2} \lesssim \frac{1}{\lambda},\ \ \|\partial_\lambda vR(H_0,\lambda^4)(\cdot,y)\|_{L^2} \lesssim \frac{1}{\lambda^2} \tag{49}$$

uniformly in y. Therefore we estimate the contribution of the error term to $\mathcal{E}(\lambda)(x,y)$ by

$$\lambda\|vR(H_0,\lambda^4)(\cdot,x)\|_{L^2}\|vR(H_0,\lambda^4)(\cdot,y)\|_{L^2} \lesssim \frac{1}{\lambda},$$

and its λ derivative by $\frac{1}{\lambda^2}$. Hence, the claim follows from Lemma 3.1 with $\alpha = 1$.

Now, consider the contribution of $Q\Gamma_{-1}Q$ to (48):

$$\big[R(H_0,\lambda^4)vQ\Gamma_{-1}QvR(H_0,\lambda^4)\big](x,y).$$

Note that, by Lemma 5.1, we bound this term by

$$\|QvR(H_0,\lambda^4)(\cdot,y)\|_{L^2}\|QvR(H_0,\lambda^4)(\cdot,x)\|_{L^2}\||\Gamma_{-1}|\|_{L^2\to L^2} \lesssim \frac{1}{\lambda}$$

uniformly in x,y. Similarly, its λ-derivative is bounded by $\frac{1}{\lambda^2}$. Therefore, the claim follows from Lemma 3.1.

The contributions of $Q\Gamma_0$ and $\Gamma_0 Q$ can be bounded similarly by using Lemma 5.1 on one side and (49) on the other side. □

Theorem 5.3. *Assume that $v(x) \lesssim \langle x\rangle^{-\frac{11}{2}-}$. If $S_2 \neq 0$, $S_3 = 0$ then*

$$\sup_{x,y\in\mathbb{R}^3}\left|\int_0^\infty e^{it\lambda^4}\lambda^3\chi(\lambda)R_V^{\pm}(\lambda^4)(x,y)\,d\lambda - F^{\pm}(x,y)\right| \lesssim \langle t\rangle^{-\frac{3}{4}}. \tag{50}$$

Here $F^{\pm}$ are time dependent finite rank operators satisfying $\|F^{\pm}\|_{L^1\to L^\infty} \lesssim \langle t\rangle^{-\frac{1}{4}}$.

Moreover if $v(x) \lesssim \langle x\rangle^{-\frac{15}{2}-}$ *and* $S_3 \neq 0$*, then*

$$\sup_{x,y\in\mathbb{R}^3}\left|\int_0^\infty e^{it\lambda^4}\lambda^3\chi(\lambda)[R_V^+(\lambda^4)-R_V^-(x,y)]\,d\lambda - G(x,y)\right| \lesssim \langle t\rangle^{-\frac{1}{2}}, \tag{51}$$

where G *is a time dependent finite rank operator satisfying* $\|G\|_{L^1\to L^\infty} \lesssim \langle t\rangle^{-\frac{1}{4}}$.

Proof. We first prove (50). By Theorem 4.4, in the case of a resonance of the second kind, we have

$$[M^\pm(\lambda)]^{-1} = S_2\Gamma_{-3}S_2 + S_2\Gamma_{-2}Q + Q\Gamma_{-2}S_2 + S_2\Gamma_{-1} + \Gamma_{-1}S_2 + Q\Gamma_{-1}Q + Q\Gamma_0 + \Gamma_0 Q + \Gamma_1.$$

We only consider the contribution of $S_2\Gamma_{-3}S_2$ to (15), the others can be handled similarly. Let

$$\mathcal{E}(\lambda,x,y) = \big[R^\pm(H_0,\lambda^4)vS_2\Gamma_{-3}S_2vR^\pm(H_0,\lambda^4)\big](x,y)$$

Note that by Lemma 5.1 we have

$$\begin{aligned}\mathcal{E} = G_0vS_2\Gamma_{-3}S_2vG_0 + G_0vS_2\Gamma_{-3}S_2v(R^\pm(H_0,\lambda^4)-G_0)\\ + (R^\pm(H_0,\lambda^4)-G_0)vS_2\Gamma_{-3}S_2vG_0 + O_1(\lambda^{-1}).\end{aligned}$$

By Lemma 3.1, the contribution of the last term is $\lesssim \langle t\rangle^{-\frac{3}{4}}$. Moreover, noting that $S_2v=0$, we have

$$\big\|[S_2vG_0](\cdot,y)\big\|_{L^2} = \Big\|\int_{\mathbb{R}^3} S_2(\cdot,y_1)v(y_1)[|y-y_1|-|y|]dy_1\Big\|_{L^2} \lesssim 1,$$

since $|[|y-y_1|-|y|]| \lesssim \langle y_1\rangle$. Therefore, the first term is $O_1(\lambda^{-3})$, and by Lemma 3.1 its contribution is $\lesssim \langle t\rangle^{-\frac{1}{4}}$. Also note that its contribution is finite rank since S_2 is. Similarly the contributions of second and third terms are $\lesssim \langle t\rangle^{-\frac{1}{2}}$, and finite rank. One can explicitly construct the operators $F^\pm(x,y)$ from the contribution of these operators to the Stone formula, (5).

Next we prove (51). Note that all the term in $M(\lambda)^{-1}$ in Theorem 4.4 except $\lambda^{-4}D_3$ are similar to the terms in the $M^{-1}(\lambda)$ that we considered in the case of resonance of the second kind. Therefore, we only control the terms interacting with D_3, that is we need to control the contribution of the following term to the Stone's formula,

$$\begin{aligned}&[R^+(H_0,\lambda^4)-R^-(H_0,\lambda^4)]v\frac{D_3}{\lambda^4}vR^+(H_0,\lambda^4)\\ &\qquad = [R^+(H_0,\lambda^4)-R^-(H_0,\lambda^4)]v\frac{D_3}{\lambda^4}vG_0\end{aligned}$$

$$+ [R^+(H_0,\lambda^4) - R^-(H_0,\lambda^4)]v\frac{D_3}{\lambda^4}v[R^+(H_0,\lambda^4) - G_0].$$

Using Lemma 5.1, the first term is $O_1(\lambda^{-3})$, and hence its contribution to Stone's formula is $\langle t\rangle^{-\frac14}$ by Lemma 3.1, and is finite rank. Similarly, the second term is $O_1(\lambda^{-2})$ and its contribution is $\lesssim \langle t\rangle^{-\frac12}$. $G(x,y)$ is obtained explicitly by inserting these operators in (5). □

We note that the time decay of the non-finite rank portion of the evolution when $S_3 \neq 0$ can be improved at the cost of spatial weights.

Corollary 5.4. *If $v(x) \lesssim \langle x\rangle^{-\frac{15}{2}-}$ and $S_3 \neq 0$, then*

$$\left| \int_0^\infty e^{it\lambda^4}\lambda^3\chi(\lambda)[R_V^+(\lambda^4) - R_V^-(x,y)]\,d\lambda - G(x,y)\right| \lesssim \langle t\rangle^{-\frac34}\langle x\rangle^{\frac52}\langle y\rangle^{\frac52}, \tag{52}$$

where G is a time dependent finite rank operator satisfying $\|G\|_{L^1\to L^\infty} \lesssim \langle t\rangle^{-\frac14}$.

Proof. We need only supply a new bound for the contribution of the following

$$[R^+(H_0,\lambda^4) - R^-(H_0,\lambda^4)]v\frac{D_3}{\lambda^4}v[R^+(H_0,\lambda^4) - G_0]. \tag{53}$$

We note that $S_3vP_2(x) = 0$ for any quadratic polynomial in the x_j variables. Hence, $S_3vG_1 = 0$ as we may write $G_1(x,y) = |x|^2 - 2x\cdot y + |y|^2$. By truncating the expansion in (10) earlier, we see

$$[R^+(H_0,\lambda^4) - R^-(H_0,\lambda^4)] = \frac{a^+ - a^-}{\lambda} + (a_1^+ - a_1^-)\lambda G_1 + O((\lambda|x-y|)^\ell|x-y|) \qquad 1<\ell\leq 3.$$

Using the orthogonality relations above and selecting $\ell = \frac32$, one can see that

$$[R^+(H_0,\lambda^4) - R^-(H_0,\lambda^4)](x,\cdot)vS_3 = O_1(\lambda^{\frac32}\langle x\rangle^{\frac52})$$

A very similar computation shows that

$$S_3v[R^+(H_0,\lambda^4) - G_0](\cdot,y) = O_1(\lambda^{\frac32}\langle y\rangle^{\frac52}).$$

Combining these, we see that

$$[R^+(H_0,\lambda^4) - R^-(H_0,\lambda^4)]v\frac{D_3}{\lambda^4}v[R^+(H_0,\lambda^4) - G_0] = O_1(\lambda^{-1}\langle x\rangle^{\frac52}\langle y\rangle^{\frac52}).$$

Applying Lemma 3.1 proves the claim.

□

6. The Perturbed Evolution For Large Energy

For completeness, we include a proof of the dispersive bound for the large energy portion of the evolution. Here we need to assume the lack of eigenvalues embedded in $[0,\infty)$ for the perturbed fourth order operator $H=(-\Delta)^2+V$. It is known that embedded eigenvalues may exist even for compactly supported smooth potentials. To complete the proof of Theorem 1.1 we show

Proposition 6.1. *Let $|V(x)|\lesssim \langle x\rangle^{-3-}$, and assume there are no embedded eigenvalues in the continuous spectrum of H, then*

$$\sup_{x,y\in\mathbb{R}^3}\left|\int_0^\infty e^{it\lambda^4}\lambda^3\widetilde{\chi}(\lambda)R_V^{\pm}(\lambda^4)(x,y)\,d\lambda\right|\lesssim |t|^{-\frac{3}{4}}. \tag{54}$$

To prove the Proposition 6.1 we use the resolvent identities and write,

$$R_V(\lambda^4)=R^{\pm}(H_0,\lambda^4)-R^{\pm}(H_0,\lambda^4)VR^{\pm}(H_0,\lambda^4)+R^{\pm}(H_0,\lambda^4)VR_V(\lambda^4)VR^{\pm}(H_0,\lambda^4). \tag{55}$$

Recall by the second part of Remark 3.3, we know that the first summand in (55) satisfies the bound in (54). Therefore, it suffices to establish the bound in Proposition 6.1 is valid for the last two summands in (55). Recall by (14), we have

$$R^{\pm}(H_0,\lambda^4)(x,y)=O_1(\lambda^{-1}). \tag{56}$$

This, along with the fact that $\lambda\gtrsim 1$, shows that

$$R^{\pm}(H_0,\lambda^4)VR^{\pm}(H_0,\lambda^4)=O_1(\lambda^{-1}),$$

as the following bounds hold uniformly in x,y:

$$|R^{\pm}(H_0,\lambda^4)VR^{\pm}(H_0,\lambda^4)(x,y)|\lesssim\lambda^{-1}\int_{\mathbb{R}^3}|V(x_1)|dx_1\lesssim\lambda^{-1}$$

$$|\partial_\lambda\{R^{\pm}(H_0,\lambda^4)VR^{\pm}(H_0,\lambda^4)]\}(x,y)|\lesssim\lambda^{-2}\int_{\mathbb{R}^3}|V(x_1)|dx_1\lesssim\lambda^{-2}.$$

Hence, by first part of Remark 3.3, $R^{\pm}VR^{\pm}$ contributes $|t|^{-\frac{3}{4}}$ to Stone's formula.

We next consider the last term in (55). To control this term, we utilize the following.

Theorem 6.2. [9, Theorem 2.23] *Let $|V(x)|\lesssim\langle x\rangle^{-k-1}$. Then for any $\sigma>k+1/2$, $\partial_z^kR_V(z)\in\mathcal{B}(L^{2,\sigma}(\mathbb{R}^d),L^{2,-\sigma}(\mathbb{R}^d))$ is continuous for $z\notin 0\cup\Sigma$. Further,*

$$\|\partial_z^kR_V(z)\|_{L^{2,\sigma}(\mathbb{R}^d)\to L^{2,-\sigma}(\mathbb{R}^d)}\lesssim z^{-(3+3k)/4}.$$

The following suffices to finish the proof of Proposition 6.1.

Lemma 6.3. *Let $|V(x)| \lesssim \langle x\rangle^{-3-}$, then*

$$\sup_{x,y\in\mathbb{R}^3}\Big|\int_0^\infty e^{-it\lambda^4}\widetilde{\chi}(\lambda)\lambda^3[R^\pm(H_0,\lambda^4)VR_V^\pm(\lambda^4)VR^\pm(H_0,\lambda^4)](x,y)d\lambda\Big| \lesssim |t|^{-\frac{3}{4}},$$

Proof. Recalling the proof of Lemma 3.1, it suffices to establish

$$\|R^\pm(H_0,\lambda^4)VR_V(\lambda^4)VR^\pm(H_0,\lambda^4)\|_{L^1\to L^\infty} \lesssim \lambda^{-1}$$
$$\|\partial_\lambda\{R^\pm(H_0,\lambda^4)VR_V(\lambda^4)VR^\pm(H_0,\lambda^4)\}\|_{L^1\to L^\infty} \lesssim \lambda^{-2}$$

Note that first by (56), and using that $L^\infty \subset L^{2,-\frac{3}{2}-}$, we have

$$\|[R^\pm(H_0,\lambda^4)]\|_{L^1\to L^{2,-\sigma}} = O_1(\lambda^{-1}), \quad \sigma > 3/2, \tag{57}$$

along with the dual estimate as an operator from $L^{2,\sigma}\to L^\infty$. Hence, by Theorem 6.2 we have the following estimate

$$\|[R^\pm(H_0,\lambda^4)VR_V(\lambda^4)VR^\pm(H_0,\lambda^4)]\|_{L^1\to L^\infty}$$
$$\lesssim \|R^\pm(H_0,\lambda^4)\|_{L^{2,\sigma}\to L^\infty}\|VR_V(\lambda^4)V\|_{L^{2,-\sigma}\to L^{2,\sigma}}\|R^\pm(H_0,\lambda^4)\|_{L^1\to L^{2,-\sigma}} \lesssim \lambda^{-1}$$

for any $|V(x)| \lesssim \langle x\rangle^{-2-}$. In fact, one can show this term is smaller, though this bound is valid since $\lambda \gtrsim 1$. Similarly, by (56) and Theorem 6.2 with $z=\lambda^4$ one obtains

$$\|\partial_\lambda\{R^\pm(H_0,\lambda^4)VR_V(\lambda^4)VR^\pm(H_0,\lambda^4)\}\|_{L^1\to L^\infty} \lesssim \lambda^{-2}$$

for any $|V(x)| \lesssim \langle x\rangle^{-3-}$. Here, we note that the extra decay on V is needed when the derivative falls on the perturbed resolvent R_V so that V maps $L^{2,-\frac{3}{2}-}\to L^{2,\frac{3}{2}+}$.

□

7. Classification of threshold spectral subspaces

In this section we establish the relationship between the spectral subspaces $S_iL^2(\mathbb{R}^3)$ for $i=1,2,3$ and distributional solutions to $H\psi=0$.

Lemma 7.1. *Assume $|v(x)| \lesssim \langle x\rangle^{-\frac{5}{2}-}$, if $\phi\in S_1L^2(\mathbb{R}^3)\setminus\{0\}$, then $\phi = Uv\psi$ where $\psi\in L^\infty$, $H\psi=0$ in distributional sense, and*

$$\psi = c_0 - G_0v\phi, \ \text{where}\ \ c_0 = \frac{1}{\|V\|_{L^1}}\langle v, T\phi\rangle. \tag{58}$$

Proof. Assume $\phi\in S_1L^2(\mathbb{R}^4)$, one has $QTQ\phi = Q(U+vG_0v)\phi = 0$. Note that

$$0 = Q(U+vG_0v)\phi = (I-P)(U+vG_0v)\phi$$
$$= U\phi + vG_0v\phi - PT\phi$$

$$\implies \phi = Uv(-G_0 v\phi + c_0) = Uv\psi \text{ where } c_0 = \frac{1}{\|V\|_{L^1}}\langle v, T\phi\rangle.$$

To show $[\Delta^2+V]\psi = [\Delta^2+V](-G_0v\phi+c_0) = 0$, notice that by differentiation under the integral sign

$$\Delta^2 G_0 v\phi = -\Delta \int \frac{1}{4\pi|x-y|} v(y)\phi(y)dy.$$

Since $\frac{1}{4\pi|x-y|}$ is the Green's function for $-\Delta$, we have $\Delta^2 G_0 v\phi = v\phi$ in the sense of distributions. Hence,

$$[\Delta^2+V](-G_0v\phi+c_0) = -v\phi + vUv\psi = 0.$$

Next, we show that $G_0 v\phi \in L^\infty$. Noting that $S_1 \leq Q$, we have $P\phi = 0$ and hence

$$\Big|\int_{\mathbb{R}^3} |x-y| v(y)\phi(y)dy\Big| = \Big|\int_{\mathbb{R}^3} [|x-y|-|x|] v(y)\phi(y)dy\Big| \lesssim \int_{\mathbb{R}^3} \langle y\rangle |v(y)\phi(y)|dy < \infty. \tag{59}$$

□

The following lemma gives further information for the function ψ in Lemma 7.1.

Lemma 7.2. *Let $|v(x)| \lesssim \langle x\rangle^{-\frac{11}{4}-}$. Let $\phi = Uv\psi \in S_1L^2(\mathbb{R}^3)\setminus\{0\}$ as in Lemma 7.1. Then,*

$$\psi = c_0 + \sum_{i=1}^{3} c_i \frac{x_i}{\langle x\rangle} + \sum_{1\leq i\leq j\leq 3} c_{ij}\frac{x_i x_j}{\langle x\rangle^3} + \widetilde{\psi}, \tag{60}$$

where $c_0 = \frac{1}{\|V\|_{L^1}}\langle v, T\phi\rangle$ and $\widetilde{\psi} \in L^2\cap L^\infty$. Moreover, $\psi \in L^p$ for $3 < p \leq \infty$ if and only if $PT\phi = 0$ and $\int yv(y)\phi(y)dy = 0$.

Furthermore, $\psi \in L^p$ for $2 \leq p \leq \infty$ if and only if $PT\phi = 0$, $\int yv(y)\phi(y)dy = 0$, and $\int y_i y_j v(y)\phi(y)dy = 0$, $1 \leq i \leq j \leq 3$.

Proof. Note that all the terms in the expansion and the function ψ are in L^∞, therefore it suffices to prove the claim for $|x| > 1$. Using Lemma 7.1 and the fact that $P\phi = 0$, we write

$$\begin{aligned}\psi(x) - c_0 &= -\frac{1}{8\pi}\int_{\mathbb{R}^3} |x-y|[v\phi](y)dy \\ &= -\frac{1}{8\pi}\int_{\mathbb{R}^3}\Big(|x-y| - |x| + \frac{x\cdot y}{|x|} + \frac{|y|^2}{2|x|} - \frac{(x\cdot y)^2}{|x|^3}\Big)[v\phi](y)dy \\ &\qquad + \frac{1}{8\pi}\int_{\mathbb{R}^3}\Big(\frac{x\cdot y}{|x|} + \frac{|y|^2}{2|x|} - \frac{(x\cdot y)^2}{|x|^3}\Big)[v\phi](y)dy =: \psi_1 + \psi_2.\end{aligned}$$

We first claim that $\psi_1\chi_{B(0,1)^c} \in L^2\cap L^\infty$. To prove this claim we first consider the case $|y| < |x|/2$. In this case, by a Taylor expansion we have

$$|x-y| = |x|\Big(1 - \frac{x\cdot y}{|x|^2} + \frac{|y|^2}{2|x|^2} - \frac{1}{8}\Big(-\frac{x\cdot y}{|x|^2} + \frac{|y|^2}{2|x|^2}\Big)^2\Big) + O(|y|^3/|x|^2) \tag{61}$$

$$= |x| - \frac{x\cdot y}{|x|} + \frac{|y|^2}{2|x|} - \frac{(x\cdot y)^2}{8|x|^3} + O\Big(\frac{|y|^{\frac52+}}{|x|^{\frac32+}}\Big)$$

Using this and the fact that $|v\phi| = |V\psi| \lesssim v^2$ we have

$$\Big|\int_{|y|<|x|/2}\Big(|x-y| - |x| + \frac{x\cdot y}{|x|} + \frac{|y|^2}{2|x|} - \frac{(x\cdot y)^2}{8|x|^3}\Big)[v\phi](y)dy\Big|$$
$$\lesssim \int_{|y|<|x|/2}\frac{|y|^{\frac52+}}{|x|^{\frac32+}}\langle y\rangle^{-\frac{11}{2}-}dy \lesssim |x|^{-\frac32-}\int_{\mathbb{R}^3}\langle y\rangle^{-3-}\,dy \lesssim |x|^{-\frac32-}$$

which belongs to $L^2\cap L^\infty$ on $B(0,1)^c$.

In the case $|y|>|x|/2$, we have

$$\Big|\int_{|y|>|x|/2}\Big(|x-y| - |x| + \frac{x\cdot y}{|x|} + \frac{|y|^2}{2|x|} - \frac{(x\cdot y)^2}{8|x|^3}\Big)[v\phi](y)dy\Big|$$
$$\lesssim \int_{|y|>|x|/2}\Big(|y| + \frac{|y|^2}{|x|}\Big)|y|^{-\frac{11}{2}-}dy \lesssim |x|^{-\frac32-},$$

which yields the claim.

Now note that for $|x|>1$

$$8\pi\psi_2 = \sum_{i=1}^3\frac{x_i}{|x|}\int_{\mathbb{R}^3}y_i[v\phi](y)dy + \frac{1}{2|x|}\int_{\mathbb{R}^3}|y|^2[v\phi](y)dy - \sum_{i,j=1}^3\frac{x_ix_j}{|x|^3}\int_{\mathbb{R}^3}y_iy_j[v\phi](y)dy. \tag{62}$$

This yields the expansion for ψ since for $|x|>1$, $\frac{x_i}{|x|} - \frac{x_i}{\langle x\rangle} = O(|x|^{-2})$ and $\frac{x_ix_j}{|x|^3} - \frac{x_ix_j}{\langle x\rangle^3} = O(|x|^{-2})$.

Noting that the second and third terms in (62) are in $L^p_{B(0,1)^c}$ for $3<p\le\infty$, we see that $\psi\in L^p$, $3<p\le\infty$, if and only if

$$c_0 + \frac{1}{8\pi}\sum_{i=1}^3\frac{x_i}{|x|}\int_{\mathbb{R}^3}y_i[v\phi](y)dy \in L^{3+}_{B(0,1)^c},$$

which is equivalent to $c_0=0$ and $\int_{\mathbb{R}^3}y_i[v\phi](y)dy=0$, $i=1,2,3$. To obtain the final claim, to determine if $\psi\in L^2_{B(0,1)^c}$ we rewrite the last two terms in (62) as follows

$$\frac{1}{|x|^3}\Big(\sum_{i=1}^3\big(\tfrac{|x|^2}{2} - x_i^2\big)\int_{\mathbb{R}^3}y_i^2[v\phi](y)dy - 2\sum_{1\le i<j\le 3}x_ix_j\int_{\mathbb{R}^3}y_iy_j[v\phi](y)dy\Big).$$

Note that the term in the parentheses is a degree 2 polynomial in x, and hence cannot be in L^2 unless all coefficients are zero, which implies the final claim. □

The following lemma is the converse of Lemma 7.1.

Lemma 7.3. *Let $|v(x)| \lesssim \langle x\rangle^{-\frac{11}{4}-}$. Assume that a nonzero function $\psi \in L^\infty$ solves $H\psi = 0$ in the sense of distributions. Then $\phi = Uv\psi \in S_1L^2$, and we have $\psi = c_0 - G_0v\phi$, $c_0 = \frac{1}{\|V\|_{L^1}}\langle v, T\phi\rangle$. In particular, the expansion given in Lemma 7.2 is valid.*

Proof. Let $\psi \in L^\infty$ be a solution of $H\psi = 0$, or equivalently $-\Delta^2\psi = V\psi$. We first show that for $\phi = Uv\psi \in QL^2$, namely

$$\int_{\mathbb{R}^3} v(x)\phi(x)dx = 0.$$

Note that $v\phi = V\psi \in L^1$. Let $\eta(x)$ be a smooth cutoff function with $\eta(x) = 1$ for all $|x| \le 1$. For $\delta > 0$, let $\eta_\delta(x) = \eta(\delta x)$. We have

$$|\langle v\phi, \eta_\delta\rangle| = |\langle V\psi, \eta_\delta\rangle| = |\langle \Delta^2\psi, \eta_\delta\rangle| = |\langle \psi, \Delta^2\eta_\delta\rangle| \le \|\psi\|_{L^\infty}\|\Delta^2\eta_\delta\|_{L^1} \lesssim \delta.$$

Therefore, taking $\delta \to 0$ and using the dominated convergence theorem we conclude that $\langle v, \phi\rangle = 0$.

Moreover, let $\tilde\psi = \psi + G_0v\phi$, then by assumption and (59), $\tilde\psi$ is bounded and $\Delta^2\tilde\psi = 0$. By Liouville's theorem for biharmonic functions on $\mathbb{R}^n$, $\tilde\psi = c$. This implies that $\psi = c - G_0v\phi$. Since

$$0 = H\psi = [\Delta^2 + V]\psi = Vc - Uv(U + vG_0v)\phi \Rightarrow v^2c = vT\phi,$$

we have $c = c_0 = \frac{1}{\|V\|_1}\langle v, T\phi\rangle$. Lastly notice that,

$$\begin{aligned} Q(U + vG_0v)Q\phi = Q(U + vG_0v)\phi &= Q(U\phi + vG_0v\phi) \\ &= Q(U\phi - v\psi + c_0v) = Q(c_0v) = 0, \end{aligned}$$

hence $\phi \in S_1L^2$ as claimed. □

Let $T_1 = S_1TPTS_1 - \frac{\|V\|_1}{3(8\pi)^2}S_1vG_1vS_1$, and S_2 be the Riesz projection on the the kernel of T_1. Moreover, let S_2' be the Riesz projection on the the kernel of S_1TPTS_1 and S_2'' be the Riesz projection on the the kernel of $S_1vG_1vS_1$.

Lemma 7.4. *Let $|v(x)| \lesssim \langle x\rangle^{-\frac{11}{4}-}$. Then, $S_2L^2 = S_2'L^2 \cap S_2''L^2$. Moreover $\int yv(y)S_2\phi(y)dy = 0$ and $PTS_2 = QvG_1vS_2 = S_2vG_1vQ = 0$. Finally, $\phi = Uv\psi \in S_1L^2$ belongs to S_2L^2 if and only if $\psi \in L^p$, $p > 3$.*

Proof. It suffices to prove that $S_2L^2 \subset S_2'L^2 \cap S_2''L^2$ since reverse inclusion holds trivially. Let $\phi \in S_1L^2$. We have

$$\langle S_1TPTS_1\phi, \phi\rangle = \langle PT\phi, PT\phi\rangle = \|PT\phi\|_2^2. \tag{63}$$

On the other hand, since $S_1 v = 0$ and x, y and v are real, we have

$$\langle S_1 v G_1 v S_1 \phi, \phi \rangle = \int_{\mathbb{R}^6} \phi(x) v(x) |x-y|^2 \overline{v(y)\phi(y)} dy dx$$

$$= \int_{\mathbb{R}^6} \phi(x) v(x) [|x|^2 - 2x \cdot y - |y|^2] \overline{v(y)\phi(y)} dy dx$$

$$= -2 \int_{\mathbb{R}^6} \phi(x) v(x) x \cdot \overline{y v(y)\phi(y)} dy dx = -2 \Big| \int_{\mathbb{R}^3} y v(y)\phi(y) dy \Big|^2 \tag{64}$$

Hence, if $\phi \in S_2 L^2$ then we have

$$0 = \langle T_1 \phi, \phi \rangle = \langle TPT\phi, \phi \rangle - \frac{\|V\|_1}{3(8\pi)^2} \langle v G_1 v \phi, \phi \rangle = \|PT\phi\|_2^2 + \frac{2\|V\|_1}{3(8\pi)^2} \Big| \int_{\mathbb{R}^3} y v(y) \phi(y) dy \Big|^2.$$

Therefore,

$$||PT\phi\|_2 = \Big| \int_{\mathbb{R}^3} y v(y) \phi(y) dy \Big| = 0,$$

which yields the claim.

This also implies that $\int y v(y) S_2 \phi(y) dy = PTS_2 = 0$ and

$$Q v G_1 v S_2 \phi = -2 Q v(x) x \cdot \int y v(y) S_2 \phi(y) dy = 0.$$

Finally, by Lemma 7.2, $\psi \in L^p$, $3 < p \le \infty$ if and only if $PT\phi = \int y v(y) \phi(y) dy = 0$, which is equivalent to $\phi \in S_2 L^2$ by the argument above. □

Define S_3 the projection on to the kernel of $T_2 = S_2 v G_3 v S_2 + \frac{10}{3} S_2 v W v S_2$, where $W(x,y) = |x|^2 |y|^2$. Note that the kernel of G_3 is

$$|x-y|^4 = |x|^4 + |y|^4 - 4x \cdot y |y|^2 - 4y \cdot x |x|^2 + 2|x|^2|y|^2 + 4(x \cdot y)^2. \tag{65}$$

Since $S_2 x_j v = S_2 v = 0$, all but the final two terms contribute zero to $S_2 v G_3 v S_2$. Therefore the kernel of T_2 (as an operator on $S_2 L^2$) is

$$T_2(x,y) = v(x) \big[\frac{26}{3} |x|^2 |y|^2 + 4(x \cdot y)^2 \big] v(y). \tag{66}$$

Lemma 7.5. *Let $|v(x)| \lesssim \langle x \rangle^{-4-}$. Fix $\phi = U v \psi \in S_2 L^2$. Then $\phi \in S_3 L^2$ if and only if $\psi \in L^p$, for all $2 \le p \le \infty$. Moreover the kernel of T_2 agrees with the kernel of $S_2 v G_3 v S_2$.*

Proof. Using (66) for $\phi \in S_2 L^2$, we have

$$\langle T_2 \phi, \phi \rangle = \frac{26}{3} \Big| \int_{\mathbb{R}^3} |y|^2 v(y) \phi(y)\, dy \Big|^2 + 4 \sum_{i,j=1}^{3} \Big| \int y_i y_j v(y) \phi(y)\, dy \Big|^2.$$

In particular, T_2 is positive semi-definite. Therefore $\phi \in S_3 L^2$, if and only if $\langle T_2 \phi, \phi \rangle = 0$, which by the calculation above equivalent to $\int y_i y_j v(y) \phi(y)\, dy = 0$ for all i, j. The claim now follows from Lemma 7.2.

The claim for $S_2vG_3vS_2$ also follows from this since by the calculation before the lemma its kernel is $v(x)\big[2|x|^2|y|^2+4(x\cdot y)^2\big]v(y)$. $\square$

Lemma 7.6. *Let $|v(x)|\lesssim\langle x\rangle^{-4-}$. Then the kernel of the operator $S_3vG_4vS_3$ on S_3L^2 is trivial.*

Proof. Take ϕ in the kernel of $S_3vG_4vS_3$. Using (1), we have (for $0<\lambda<1$)

$$R(H_0;-\lambda^4)=\frac{1}{2i\lambda^2}\big[R_0(i\lambda^2)-R_0(-i\lambda^2)\big]=\frac{e^{i\sqrt{i}\lambda|x-y|}-e^{i\sqrt{-i}\lambda|x-y|}}{8\pi i\lambda^2|x-y|}.$$

By an expansion similar to (10), and the proof of Lemma 4.1, we have for $0<\lambda<1$ and for all $|x-y|$,

$$R(H_0;-\lambda^4)=\frac{a_0}{\lambda}+G_0+a_1\lambda G_1+a_3\lambda^3G_3+a_4\lambda^4G_4+O(|\lambda|^{4+}|x-y|^{5+}),$$

where $a_0,a_1,a_3,a_4\in\mathbb{C}$ are constants. Notice that since $\phi\in S_3L^2$ one has $0=\langle v,\phi\rangle=\langle G_1v\phi,v\phi\rangle=\langle G_3v\phi,v\phi\rangle$. Also note that since $v\phi=V\psi$, we have

$$\iint|x-y|^{5+}v(x)v(y)|\phi(x)\phi(y)|dxdy\lesssim\iint|x-y|^{5+}\langle x\rangle^{-8-}\langle y\rangle^{-8-}dxdy<\infty.$$

Therefore

$$\begin{aligned}(67)\qquad 0&=\langle S_3vG_4v\phi,\phi\rangle=\langle G_4v\phi,v\phi\rangle\\&=\lim_{\lambda\to0}\Big\langle\frac{R(H_0;-\lambda^4)-a_0\lambda^{-1}-G_0-a_1\lambda G_1-a_3\lambda^3G_3}{\lambda^4}v\phi,v\phi\Big\rangle\\&=\lim_{\lambda\to0}\Big\langle\frac{R(H_0;-\lambda^4)-G_0}{\lambda^4}v\phi,v\phi\Big\rangle.\end{aligned}$$

Further, recalling that $G_0=[\Delta^2]^{-1}$ and considering the Fourier domain, one has

$$\begin{aligned}(68)\quad 0&=\lim_{\lambda\to0}\Big\langle\frac{R(H_0;-\lambda^4)-G_0}{\lambda^4}v\phi,v\phi\Big\rangle\\&=\lim_{\lambda\to0}\frac{1}{\lambda^4}\Big\langle\Big(\frac{1}{8\pi^2\xi^4+\lambda^4}-\frac{1}{8\pi^2\xi^4}\Big)\widehat{v\phi}(\xi),\widehat{v\phi}(\xi)\Big\rangle\\&=\lim_{\lambda\to0}\int_{\mathbb{R}^3}\frac{-1}{(8\pi^2\xi^4+\lambda^4)8\pi^2\xi^4}|\widehat{v\phi}(\xi)|^2d\xi=\frac{-1}{64\pi^4}\int_{\mathbb{R}^3}\frac{|\widehat{v\phi}(\xi)|^2}{\xi^8}d\xi.\end{aligned}$$

Where we used the Monotone Convergence Theorem in the last step.

Note that this gives $v\phi=0$ since $v\phi\in L^1$. Also noting that the support of $\phi=Uv\psi$ is a subset of the support of v, we have $\phi=0$. This establishes the invertibility of $S_3vG_4vS_3$ on S_3L^2. $\square$

Remark 7.7. *Notice that,* (67) *and* (68) *imply that for any* $\phi \in S_3$ *one has*

$$\langle S_3 v G_4 v \phi, \phi \rangle = \frac{-1}{64\pi^4} \int_{\mathbb{R}^3} \Big\langle \frac{\widehat{v\phi}(\xi)}{\xi^4}, \frac{\widehat{v\phi}(\xi)}{\xi^4} \Big\rangle = -\langle G_0 v\phi, G_0 v\phi \rangle$$

provided $|v(x)| \lesssim \langle x \rangle^{-4-}$.

Lemma 7.8. *The operator* $P_0 := G_0 v S_3 [S_3 v G_4 v S_3]^{-1} S_3 v G_0$ *is the orthogonal projection on* L^2 *onto the zero energy eigenspace of* $H = \Delta^2 + V$.

Proof. Let $\{\phi_k\}_{k=1}^N$ be the orthonormal basis of $S_3 L^2$, then $S_3 f = \sum_{j=1}^N \phi_j \langle f, \phi_j \rangle$. Moreover, for all ϕ_k, one has $\psi_k = -G_0 v \phi_k = -G_0 V \psi_k$ are linearly independent for each k and $\psi_k \in L^2$. We will show that $P_0 \psi_k = G_0 v S_3 [S_3 v G_4 v S_3]^{-1} S_3 v G_0 \psi_k = \psi_k$ for all $1 \leq k \leq N$. This implies that P_0 is the identity on the range of P_0. Since P_0 is self-adjoint, this finishes the proof.

Let $\{A_{ij}\}_{i,j=1}^N$ be the matrix that representation of $S_3 v G_4 v S_3$ with respect to the orthonormal basis $\{\phi_k\}_{k=1}^N$, then by Remark 7.7

$$A_{ij} = \langle S_3 v G_4 v \phi_j, \phi_i \rangle = -\langle G_0 v \phi_j, G_0 v \phi_i \rangle = -\langle \psi_j, \psi_i \rangle.$$

Also note that, by the representation of S_3, we have

$$S_3 v G_0 \psi_k = \sum_{j=1}^N \phi_j \langle v G_0 \psi_k, \phi_j \rangle = -\sum_{j=1}^N \phi_j \langle \psi_k, \psi_j \rangle = -\sum_{j=1}^N \phi_j A_{jk} \tag{69}$$

By (69) we have

$$P_0 \psi_k = -\sum_{j=1}^N G_0 v S_3 [S_3 v G_4 v S_3]^{-1} \phi_j A_{jk}$$
$$= -\sum_{i,j=1}^N G_0 v S_3 (A^{-1})_{ij} \phi_i A_{jk} = \sum_{i,j=1}^N \psi_i (A^{-1})_{i,j} A_{jk} = \sum_{i=1}^N \psi_i \delta_{ik} = \psi_k.$$

□

Remark 7.9. *One consequence of the preceeding results is that any zero-energy resonance function is of the form:*

$$\psi(x) = c_0 + c_1 \frac{x_1}{\langle x \rangle} + c_2 \frac{x_2}{\langle x \rangle} + c_3 \frac{x_3}{\langle x \rangle} + \sum_{1 \leq i \leq j \leq 3} c_{ij} \frac{x_i x_j}{\langle x \rangle^3} + O_{L^2}(1).$$

For some constants c_0, c_1, c_2, c_3, *and* c_{ij}, $1 \leq i \leq j \leq 3$. *Hence, the resonance space is at most 10 dimensional along with a finite-dimensional eigenspace. Moreover,* $S_1 - S_2$ *is at most four dimensional,* $S_2 - S_3$ *is at most 6 dimensional, the rest is the eigenspace.*

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